1. Maps between real vector spaces

- **Scalar field**: maps $\mathbb{R}^n \to \mathbb{R}$.
- **Vector field**: maps $\mathbb{R}^n \to \mathbb{R}^n$.
- **Curve**: maps $\mathbb{R} \to \mathbb{R}^n$.
- A **tangent** to a curve $\underline{x}(t)$ is given by $\frac{dx}{dt}$.
- The **arc-length** parameterisation of a curve $\underline{x}$ is such that

$$\left|\frac{d\underline{x}(s)}{ds}\right| = 1 \quad \forall s$$

- **Partial derivatives**:

$$\frac{\partial f(\underline{x})}{\partial x_a} = \lim_{h \to 0} \frac{f\left(\underline{x} + h\underline{e}_a\right) - f(\underline{x})}{h}$$

- **Chain rule**: for a scalar field $f(\underline{x})$ and curve $\underline{x}(t) = x_1(t)\underline{e}_1 + ... + x_n(t)\underline{e}_n$,

$$\frac{df(\underline{x}(t))}{dt} = \sum_{i=1}^{n} \frac{\partial f(\underline{x})}{\partial x_i} \frac{dx_i}{dt}$$

  Here $F(t) := f(\underline{x}(t))$ is the restriction of $f(\underline{x})$ to the curve $\underline{x}(t)$.

2. The gradient of a scalar field

- **Differential operator**: maps functions to functions, e.g.

$$\frac{d}{dt} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{dx_i}{dt}$$

- Let $f, g : \mathbb{R} \to \mathbb{R}$, then
  - $f(x)\frac{d}{dx}$ is a differential operator. It acts on $g(x)$ to give $f(x)\frac{dg(x)}{dx}$.
  - $\frac{d}{dx}f(x)$ is a differential operator. It acts on $g(x)$ to give $\frac{d}{dx}(f(x)g(x))$.
  - $\left(\frac{d}{dx}f(x)\right)$ is an differential operator. It acts on $g(x)$ to give $\frac{df(x)}{dx}g(x)$.
- **del (or nabla)**: $\underline{\nabla} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\underline{e}_i$ so $\frac{d}{dt} = \underline{\nabla}. \frac{d\underline{x}(t)}{dt}$.
- **gradient** of a scalar field $f : \mathbb{R}^n \to \mathbb{R}$:

$$\underline{\nabla} f \equiv \text{grad}(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\underline{e}_i$$

- **Directional derivative** of $f : \mathbb{R}^n \to \mathbb{R}$ in direction of a unit tangent $\underline{\hat{n}} = \frac{d\underline{x}(s)}{ds}$ to a curve $x : \mathbb{R} \to \mathbb{R}^n$:

$$\frac{df(\underline{x}(s))}{ds} = \underline{\hat{n}}. \underline{\nabla} f \equiv \frac{df}{d\underline{\hat{n}}}$$

  where $\underline{x}$ is parameterised in terms of arc-length s.
- $\underline{\nabla} f$ at a point $\underline{p}$ is orthogonal to curves contained in level set of f at $\underline{p}$.
- $\underline{\nabla} f$ points in the direction where f increases fastest.
- **Properties of the gradient**: let $f, g : \mathbb{R}^n \to \mathbb{R}$, $a, b \in \mathbb{R}$, $\varphi : \mathbb{R} \to \mathbb{R}$, then
  - $\underline{\nabla}(af + bg) = a\underline{\nabla} f + b\underline{\nabla} g$

- $\underline{\nabla}(fg) = f\underline{\nabla}g + g\underline{\nabla}f$
- $\underline{\nabla}\varphi(f) = (\underline{\nabla}f)\frac{d\varphi}{df}$

3. $\underline{\nabla}$ acting on vector fields

- **Divergence** of a vector field $\underline{v}(\underline{x}) = \sum_{i=1}^{n} v_i(\underline{x})\underline{e}_i$:

$$\underline{\nabla}.\,\underline{v} \equiv \mathrm{div}(\underline{v}) = \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i}$$

  Note that the formula will be different in other coordinates systems. Also $\underline{\nabla}.\,\underline{v} \neq \underline{\nabla} \cdot \underline{v}$.
- Considering a vector field as a fluid, if the divergence at a point is positive the vector field acts as a **source** at that point (more fluid leaving than entering), if the divergence is negative the vector field acts as a **sink** at that point (more fluid entering than leaving). The magnitude of vector at point is the rate of flow at that point and direction of vector is direction of flow.
- **Properties of div**: for $f : \mathbb{R}^n \to \mathbb{R}$, $\underline{v}, \underline{w} : \mathbb{R}^n \to \mathbb{R}^n$, $a, b \in \mathbb{R}$,
  - $\underline{\nabla}.\,(a\underline{v} + b\underline{w}) = a\underline{\nabla}.\,\underline{v} + b\underline{\nabla}.\,\underline{w}$
  - $\underline{\nabla}.\,(f\underline{v}) = (\underline{\nabla}f).\,\underline{v} + f\underline{\nabla}.\,\underline{v}$
- **Curl** of $\underline{v} : \mathbb{R}^n \to \mathbb{R}^n$:

$$\underline{\nabla} \times \underline{v} \equiv \mathrm{curl}(\underline{v}) = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \underline{e}_1\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \underline{e}_2\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \underline{e}_3\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

- Considering a vector field as a fluid, the magnitude of the curl at a point corresponds to the rotational speed of the fluid, and the direction of the curl corresponds to which axis the fluid is rotating around, determined using the right-hand rule (fingers represent rotation of the fluid, thumb points in direction of curl).
- **Properties of curl**: for $f : \mathbb{R}^3 \to \mathbb{R}$, $\underline{v}, \underline{w} : \mathbb{R}^3 \to \mathbb{R}^3$, $a, b \in \mathbb{R}$,
  - $\underline{\nabla} \times (a\underline{v} + b\underline{w}) = a\underline{\nabla} \times \underline{v} + b\underline{\nabla} \times \underline{w}$
  - $\underline{\nabla} \times (f\underline{v}) = (\underline{\nabla}f) \times \underline{v} + f\underline{\nabla} \times \underline{v}$
- **Laplacian** of $f : \mathbb{R}^n \to \mathbb{R}$:

$$\Delta f \equiv \underline{\nabla}^2 f := \underline{\nabla}.\,(\underline{\nabla}f) = \mathrm{div}(\mathrm{grad}(f)) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

  Note this formula is only valid for **cartesian** coordinates.

4. Index notation

- **Einstein summation convention**: in an expression involving a summation, then index of summation always appears twice. The convention is that the summation sign is removed, and whenever an index appears twice, it is summed over.
- **Dummy indices**: repeated indices. They can be renamed without changing the expression.

- **Free indices**: non-repeated indices. They must match on both sides of an equation.
- An index can't be repeated more than twice in the same term, so $(\underline{u}.\,\underline{v})^2 = u_i v_i u_j v_j \neq u_i v_i u_i v_i$.
- **Kronecker delta**:

$$\delta_{ij} := \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \qquad = \frac{\partial x_i}{\partial x_j}$$

- If δ_{ij} has a dummy index i, then remove the δ_{ij} and replace the dummy index i by j in the rest of the expression.
- **Levi-Cevita symbol**:

$$\varepsilon_{ijk} = -\varepsilon_{jik} = -\varepsilon_{ikj} \quad \text{(antisymmetry)}\ \varepsilon_{123} = 1$$

- **Properties of ε_{ijk}**:
  - $\varepsilon_{ijk} = -\varepsilon_{kji}$
  - $\varepsilon_{ijk} = 0$ if $i = j$ or $j = k$ or $k = i$
  - If $\varepsilon_{ijk} \neq 0$ then $(i\ j\ k)$ is a permutation of $(1\ 2\ 3)$.
  - $\varepsilon_{ijk} = 1$ if $(i\ j\ k)$ is an even permutation of $(1\ 2\ 3)$ (even number of swaps).
  - $\varepsilon_{ijk} = -1$ if $(i\ j\ k)$ is an odd permutation of $(1\ 2\ 3)$ (odd number of swaps).
  - $\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$ (cyclic permutation).
- The cross product $\underline{C} = \underline{A} \times \underline{B}$ can be written as $C_i = \varepsilon_{ijk} A_j B_k$.
- **Very useful ε_{ijk} formula**:

$$\sum_{k=1}^{3} \varepsilon_{ijk} \varepsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

- Notation: $\partial_i := \frac{\partial}{\partial x_i}$.
- $\underline{\nabla}.\,\underline{v} = \frac{\partial v_i}{\partial x_i} = \partial_i v_i$.
- $(\underline{\nabla} \times \underline{v})_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} v_k = \varepsilon_{ijk} \partial_j v_k$.

5. Differentiability of scalar fields

- **$f(\underline{x})$ tends to L as x tends to a**:

$$\lim_{\underline{x} \to \underline{a}} f(\underline{x}) = L \iff \forall \varepsilon > 0, \exists \delta > 0, \forall \underline{x}, 0 < |\underline{x} - \underline{a}| < \delta \implies |f(\underline{x}) - L| < \varepsilon$$

- Scalar field f **continuous** at $\underline{a}$ if $\lim_{\underline{x} \to \underline{a}} f(\underline{x})$ exists and is equal to $f(\underline{a})$
- If f and g are continuous scalar fields at $\underline{a}$ then so are:
  - $f + g$
  - fg
  - $f\ /\ g$ (if $g(\underline{a}) \neq 0$)
- $f(\underline{x}) = c$ for a constant c is continuous at every $\underline{x} \in \mathbb{R}^n$
- $f(\underline{x}) = x_a$, $a \in \{1, ..., n\}$ is continuous at every $\underline{x} \in \mathbb{R}^n$
- **Open ball, centre $\underline{a}$, radius $\delta > 0$**:

$$B_\delta(\underline{a}) := \{\underline{x} \in \mathbb{R}^n : |\underline{x} - \underline{a}| < \delta\}$$

- $S \subseteq \mathbb{R}^n$ **open** if $\forall \underline{a} \in S$, $\exists \delta > 0$ such that $B_\delta(\underline{a}) \subseteq S$

- **Neighbourhood** $N \subseteq \mathbb{R}^n$ of $\underline{a} \in \mathbb{R}^n$: contains an open set containing $\underline{a}$
- $S \subseteq \mathbb{R}^n$ **closed** if its complement $\mathbb{R}^n - S$ is open
- Every open ball is open
- Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$. f is **continuous on U** if it is continuous at every $\underline{a} \in U$
- Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$. f is **differentiable** at $\underline{a} \in U$ if for some vector $\underline{v}(\underline{a})$,

$$f(\underline{a} + \underline{h}) - f(\underline{a}) = \underline{h}.\, \underline{v}(\underline{a}) + R(\underline{h}), \quad \lim_{\underline{h} \to \underline{0}} \frac{R(\underline{h})}{|\underline{h}|} = 0$$

  If $\underline{v}(\underline{a})$ exists, $\underline{v}(\underline{a}) = \underline{\nabla} f$
- **Warning**: $\underline{\nabla} f$ being defined at a point does not imply that f is differentiable at that point.
- Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$. Then f is differentiable at $\underline{a}$ if all partial derivatives of f exist and are continuous in a neighbourhood of $\underline{a}$
- Function is **continuously differentiable** at $\underline{a}$ if it and all its partial derivatives exist and are continuous at $\underline{a}$. It is **continuously differentiable** on an open U if it and all its partial derivatives exist and are continuous on U.
- Continuous differentiability implies differentiability.
- **Smooth function**: partial derivatives of all orders exist.
- Let $U \subseteq \mathbb{R}^n$ be open. If $f, g : U \to \mathbb{R}$ differentiable (or smooth) at $\underline{a} \in \mathbb{R}^n$ then so are:
  - $f + g$
  - fg
  - $f\ /\ g$ (if $g(\underline{a}) \neq 0$)
- Let $U \subseteq \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ be differentiable, $\underline{x}$ be a function of $u_1, ...u_m$ where all partial derivatives $\frac{\partial x_i}{\partial u_j}$ exist. Let $F(u_1, ...u_m) = f(\underline{x}(u_1, ...u_m))$, then

$$\frac{\partial F}{\partial u_b} = \frac{\partial \underline{x}}{\partial u_b}.\, \underline{\nabla} f$$

- **Level set** of $f : U \to \mathbb{R}$, $U \subseteq \mathbb{R}^n$ open, is the set $\{\underline{x} \in U : f(\underline{x}) = c\}$ for some $c \in \mathbb{R}$. For $n = 2$ it is called a **level curve**.
- **Implicit function theorem for level curves**: if $f : U \to \mathbb{R}$ is differentiable, and $(x_0, y_0) \in U$ is on the level curve $f(x, y) = c$ where $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$, then there exists a differentiable function $g(x)$ in a neighbourhood of x_0 satisfying

$$g(x_0) = y_0$$
$$f(x, g(x)) = c$$
$$\frac{dg}{dx} = -\frac{\frac{\partial f(x,g(x))}{\partial x}}{\frac{\partial f(x,g(x))}{\partial y}}$$

- **Critical point**: point of level curve $f(x, y) = c$ where $\underline{\nabla} f = \underline{0}$. c is a **critical value**, otherwise it is a **regular value**.

- At a critical point, the level curve can't be written as either $y = g(x)$ or as $x = h(y)$ in a neighbourhood of Q, with g, h differentiable.
- **Implicit function theorem for level surfaces**: Let $f : U \to \mathbb{R}$ be differentiable, $U \subseteq \mathbb{R}^3$ open, $(x_0, y_0, z_0) \in U$ be on the level set $f(x, y, z) = c$. If $\frac{\partial f}{\partial z}(x_0, y_0, z_0) \neq 0$ then $f(x, y, z) = c$ defines a surface $z = g(x, y)$ in a neighbourhood of (x_0, y_0, z_0), where

$$f(x, y, g(x, y)) = c$$
$$g(x_0, y_0) = z_0$$
$$\frac{\partial g}{\partial x}(x_0, y_0) = -\frac{\frac{\partial f}{\partial x}(x_0, y_0, z_0)}{\frac{\partial f}{\partial z}(x_0, y_0, z_0)}$$
$$\frac{\partial g}{\partial y}(x_0, y_0) = -\frac{\frac{\partial f}{\partial y}(x_0, y_0, z_0)}{\frac{\partial f}{\partial z}(x_0, y_0, z_0)}$$

- $\underline{\nabla} f(x_0, y_0, z_0)$ is normal to the tangent plane of the level set $z = g(x, y)$ at (x_0, y_0). So the normal line is given by

$$\underline{x}(t) = \underline{x_0} + t\underline{\nabla} f$$

and the tangent plane is given by

$$\left(\underline{x} - \underline{x_0}\right). \underline{\nabla} f = 0$$

6. Differentiability of vector fields

- **Jacobian matrix (differential)** of $\underline{F}(\underline{x})$ at $\underline{x} = \underline{a}$ (written $D\underline{F}(\underline{a})$ or $D\underline{F}_{\underline{a}}$): matrix with components $a_{i,j} = \frac{\partial F_i}{\partial x_j}$.
- For open $U \subseteq \mathbb{R}^n$, $\underline{F} : U \to \mathbb{R}^n$ **differentiable** at $\underline{a} \in U$ if for some **linear** function $\underline{L} : \mathbb{R}^n \to \mathbb{R}^n$,

$$\underline{F}(\underline{a} + \underline{h}) - \underline{F}(\underline{a}) = \underline{L}(\underline{h}) + R(\underline{h})$$

where

$$\lim_{\underline{h} \to \underline{0}} \frac{R(\underline{h})}{|\underline{h}|} = \underline{0}$$

Here, $\underline{L}(\underline{h}) = (D\underline{F}(\underline{a}))\underline{h}$.
- **Jacobian, $J(\underline{v})$**: determinant of differential: $J(\underline{v}) = \det(D\underline{v})$
- Can think of vector fields as **coordinate transformations** on $\mathbb{R}^n$.
- **Inverse function theorem**: let U open, $v : U \to \mathbb{R}^n$ differentiable with continuous partial derivatives. If $J(\underline{v}(\underline{a})) \neq 0$ then exists open $\tilde{U} \subseteq U$ containing $\underline{a}$ such that:
  - $\underline{v}\left(\tilde{U}\right)$ is open and
  - Mapping $\underline{v}$ from $\tilde{U}$ to $\underline{v}\left(\tilde{U}\right)$ has differentiable inverse $\underline{w} : \underline{v}\left(\tilde{U}\right) \to \mathbb{R}^n$ with $\underline{v}(\underline{w}(\underline{x})) = \underline{x}$ and $\underline{w}\left(\underline{v}\left(\underline{y}\right)\right) = \underline{y}$.

- Map $\underline{v} : \tilde{U} \to V \subseteq \mathbb{R}^n$ which satisfies above two properties is called **diffeomorphism** of $\tilde{U}$ onto $\tilde{V} = \underline{v}(\tilde{U})$. $\tilde{U}$ and $\tilde{V}$ are **diffeomorphic**.
- **Local diffeomorphism**: map $\underline{v} : U \to V$ where $\forall \underline{a} \in U$, exists open $\tilde{U} \subseteq U$ containing $\underline{a}$ such that $\underline{v} : \tilde{U} \to \underline{v}(\tilde{U})$ is diffeomorphism.
- **Chain rule for vector fields**:

$$D\underline{w}(\underline{v}(\underline{x})) = D\underline{w}(\underline{v})D\underline{v}(\underline{x})$$

- When $\underline{v}$ is local diffeomorphism and $\underline{w}$ is its inverse, then

$$(D\underline{v})^{-1} = D\underline{w}, \quad J(\underline{w}) = \frac{1}{J(\underline{v})}, \quad J(\underline{v}) \neq 0$$

- $\underline{v}$ is **orientation preserving** if $J(\underline{v}) > 0$.
- $\underline{v}$ is **orientation reversing** if $J(\underline{v}) < 0$.

7. Volume, line and surface integrals

- **One dimensional integral**: calculates area under curve.

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i^*)\Delta x_i$$

  where $[a,b]$ partitioned as $a = x_0 < x_1 <\cdots< x_n = b$, $\Delta x_i = x_{i+1} - x_i$, $x_i^* \in [x_i, x_{i+1}]$ is arbitrary.
- **Double integral**: calculates volume under surface $z = f(x,y)$ over region R.

$$\int_R f(x,y)\,\mathrm{d}A = \lim_{N\to\infty} \sum_{k=1}^{N} f(x_k^*, y_k^*)\Delta A_k$$

  R is split into N rectangle ΔA_k. (x_k^*, y_k^*) lies in base of kth prism.
- If rectangles chosen on rectangular grid, then $\Delta A_k = \Delta x_i \Delta y_j$ where $\Delta x_i = x_{i+1} - x_i$, $\Delta y_j = y_{j+1} - y_j$, x and y partitioned as $x_0 < x_1 <\cdots< x_n$ and $y_0 < y_1 <\cdots< y_m$. As before $x_i^* \in [x_i, x_{i+1}]$ and $y_j^* \in [y_j, y_{j+1}]$. Integral is

$$\int_R f(x,y)\,\mathrm{d}A = \lim_{n,m\to\infty} \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(x_k^*, y_k^*)\Delta x_i \Delta y_j = \int_x \left(\int_y f(x,y)\,\mathrm{d}y \right) \mathrm{d}x$$

- **Fubini's theorem**: if $f(x,y)$ continuous over compact (bounded and closed) region A, then double integral over A can be written as **iterated integral**, with integrals in either order:

$$\int_A f(x,y)\,\mathrm{d}A = \int_y \int_x f(x,y)\,\mathrm{d}x\,\mathrm{d}x = \int_x \int_y f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

- **Important**: Fubini's theorem holds if region and/or function is unbounded, provided **double integral absolutely convergent** (integral of $|f(x,y)|$ over A is finite).
- To **calculate area in plane** (e.g. between two curves), set $f(x,y) = 1$:

$$\text{Area of } R = \int_R 1 \, \mathrm{d}A$$

- **Volume integral**: partition volume V into N volumes ΔV_i.

$$I = \int_V f(\underline{x}) \, \mathrm{d}V = \lim_{N \to \infty} \sum_{i=1}^{N} f(x_i) \Delta V_i$$

- If $f(\underline{x})$ is density of a quantity, then $I = \int_V f(\underline{x}) \, \mathrm{d}V$ is amount of that quantity.
- To **calculate volume inside surface**, (S is surface which encloses V) set $f(x, y, z) = 1$:

$$\text{Volume inside } S = \text{Volume of } V = \int_V 1 \, \mathrm{d}V$$

- As for double integrals, if V partition parallel to coordinate planes than

$$I = \int_x \int_y \int_z f(x, y, z) \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x$$

- Fubini's theorem holds for triple integrals.
- **Regular arc**: curve $\underline{x}(t)$ where $x_a(t)$ continuous with continuous first derivatives.
- **Regular curve**: finite number of regular arcs joined end to end.
- **Line integral** of $\underline{v}(\underline{x})$ along arc $C : t \to \underline{x}(t)$, $t \in [\alpha, \beta]$:

$$\int_C \underline{v} \cdot d\underline{x} = \int_\alpha^\beta \underline{v}(\underline{x}(t)) \cdot \frac{d\underline{x}(t)}{dt} \, \mathrm{d}t$$

- Line integral doesn't depend on parameterisation of C.
- Line integral along regular curve C is sum of line integrals of arcs of C. If C is closed, written $\oint_C \underline{v} \cdot \mathrm{d}\underline{x}$.
- **Length of curve**:

$$\int_C \mathrm{d}s = \int_a^b \left\| \frac{d\underline{x}(t)}{dt} \right\| \mathrm{d}t$$

- If f is density function, mass is

$$\int_C f \, \mathrm{d}s = \int_a^b f(\underline{x}(t)) \left\| \frac{d\underline{x}(t)}{dt} \right\| \mathrm{d}t$$

- If $\underline{F}$ is force, work done is

$$\int_C \underline{F} \cdot \mathrm{d}\underline{x}$$

- If curve is ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, can parameterise as $x(t) = a\cos(t)$, $y(t) = b\sin(t)$.
- If curve is $y = f(x)$, can parameterise as $x(t) = t$, $y(t) = f(t)$.
- If curve is $x = g(y)$, can parameterise as $x = g(t)$, $y(t) = t$.
- If curve is straight line segment from (x_0, y_0) to (x_1, y_1), can parameterise as $x(t) = (1 - t)x_0 + tx_1$, $y(t) = (1 - t)y_0 + ty_1$.

- Surface can be given in **parametric form** as $\underline{x}(u, v)$ where $u, v \in U$ (U is **parameter domain**).
- If curve is $z = f(x, y)$, can parameterise as $x = u, y = v, z = f(u, v)$. Similarly for $y = g(x, z)$ and $x = h(y, z)$.
- For surface S as $\underline{x}(u, v)$, **unit normal** vector is

$$\hat{\underline{n}} = \frac{\underline{a}}{|\underline{a}|}, \quad \underline{a} = \left(\frac{\partial \underline{x}(u, v)}{\partial u} \times \frac{\partial \underline{x}(u, v)}{\partial v}\right)$$

(negative of this is also).
- For surface given as **level surface** of scalar field f, $f(x, y, z) = c$, **unit normal** vector is

$$\hat{\underline{n}} = \frac{\nabla f}{|\nabla f|}$$

(negative of this is also).
- Surface $\underline{x}(u, v)$ **orientable** if partial derivatives of $\underline{x}$ exist and are continuous, and $\hat{\underline{n}}$ is continuous.
- **Surface integral** defined as

$$\int_S \underline{F} \cdot \mathrm{d}\underline{A} = \lim_{\Delta A_k \to 0} \sum_k \underline{F}\left(\underline{x}_k^*\right) \cdot \hat{\underline{n}}_k \Delta A_k$$

- For surface $\underline{x}(u, v)$,

$$\int_S \underline{F} \cdot \mathrm{d}\underline{A} = \int_U \underline{F}(\underline{x}(u, v)) \cdot \left(\frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v}\right) \mathrm{d}u \, \mathrm{d}v$$

since $\left(\frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v}\right)$ is normal to surface.
- For surface $f(x, y, z) = c$,

$$\int_S \underline{F} \cdot \mathrm{d}\underline{A} = \int_A \frac{\underline{F} \cdot \nabla f}{\underline{e}_3 \cdot \nabla f} \mathrm{d}x \, \mathrm{d}y$$

where (x, y) range over A, A is **projection** of S onto x, y plane.
- If unit normal to surface S, $\hat{\underline{n}}$, is known and $\underline{F} \cdot \hat{\underline{n}}$ is constant, then

$$\int_S \underline{F} \cdot \mathrm{d}\underline{A} = \int_S \underline{F} \cdot \hat{\underline{n}} \, \mathrm{d}A = \underline{F} \cdot \hat{\underline{n}} \int_S \mathrm{d}A = \underline{F} \cdot \hat{\underline{n}} \times \text{area of } S$$

8. Green's, Stoke's and divergence theorems

- **Green's theorem**: let $P(x, y)$ and $Q(x, y)$ be continuously differentiable scalar fields in 2 dimensions. Then

$$\oint_C (P(x, y) \, \mathrm{d}x + Q(x, y) \, \mathrm{d}y) = \int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x \, \mathrm{d}y$$

where C is boundary of A traversed in positive (anticlockwise) direction (imagine walking around C with A to your left).

- **Green's theorem in vector form**: let $\underline{F}(x, y, z) = (P(x, y), Q(x, y), R)$, then

$$\oint_C \underline{F} \cdot \mathrm{d}\underline{x} = \int_A (\underline{\nabla} \times \underline{F}) \cdot \underline{e}_3 \, \mathrm{d}A$$

- **Stokes' theorem**: let $\underline{F}(x, y, z)$ be continuously differentiable vector field, S in $\mathbb{R}^3$ be surface with area elements $\mathrm{d}\underline{A} = \hat{\underline{n}} \, \mathrm{d}A$ and boundary curve $C = \partial S$, then

$$\oint_C \underline{F} \cdot \mathrm{d}\underline{x} = \int_S (\underline{\nabla} \times \underline{F}) \cdot \mathrm{d}\underline{A}$$

  Orientation of C and choice of $\hat{\underline{n}}$ or $-\hat{\underline{n}}$ given by **right hand rule**: curl fingers of right hand and extend thumb. When thumb points in direction of surface normal, fingers point in direction of orientation of boundary, and vice versa. (Equivalently, if you stood on boundary with head pointing in direction of normal, and walked around boundary with surface on your left, direction of walking is direction of orientation of boundary.)
- **Divergence theorem**: let $\underline{F}$ be continuously differentiable vector field defined over volume V with bounding surface S, then

$$\int_S \underline{F} \cdot \mathrm{d}\underline{A} = \int_V \underline{\nabla} \cdot \underline{F} \, \mathrm{d}V$$

  where $\mathrm{d}\underline{A} = \hat{\underline{n}} \, \mathrm{d}A$, $\hat{\underline{n}}$ is outward unit normal.
- Vector field **conservative** if line integral is path independent.
- $\underline{F}$ **closed** if $\underline{\nabla} \times \underline{F} = \underline{0}$.
- Region D **simply connected** if any closed curve in D can be continuously shrunk to point in D.
- Every closed curve in D is boundary of surface in D.
- Let $\underline{F}$ vector field and $\underline{\nabla} \times \underline{F} = \underline{0}$ in simply connected region D. If C_1 and C_2 are paths in D joining $\underline{a}$ to $\underline{b}$ then

$$\int_{C_1} \underline{F} \cdot \mathrm{d}\underline{x} = \int_{C_2} \underline{F} \cdot \mathrm{d}\underline{x}$$

  so line integral is path-independent and $\underline{F}$ is conservative.
- If $\underline{F} = \underline{\nabla}\varphi$ for scalar field φ ($\underline{F}$ is **exact**) then $\int_C \underline{F} \cdot \mathrm{d}\underline{x}$ is path-independent so $\underline{F}$ is conservative. If C goes from $\underline{a}$ to $\underline{b}$ then

$$\int_C \underline{F} \cdot \mathrm{d}\underline{x} = \varphi(\underline{b}) - \varphi(\underline{a})$$

- $\underline{\nabla} \times \underline{F} = 0 \Longleftrightarrow$ path indepence of integral $\Longleftrightarrow \exists \varphi, \underline{F} = \underline{\nabla}\varphi$

9. Non-Cartesian systems

- Polar, spherical polar and cylindrical polar coordinates are all **curvilinear coordinates**.

- Cartesian coordinates (x, y, z) can be expressed as smooth functions of curvlinear coordinates (u, v, w):

$$x = g(u, v, w), y = h(u, v, w), z = k(u, v, w), \quad g, h, k \in C^\infty(\mathbb{R}^3)$$

g, h, k can be inverted to give

$$u = \tilde{g}(x, y, z), v = \tilde{h}(x, y, z), w = \tilde{k}(x, y, z), \quad \tilde{g}, \tilde{h}, \tilde{k} \in C^\infty(\mathbb{R}^3)$$

- **Coordinate surfaces**: planes with equations $u =$ constant, $v =$ constant or $w =$ constant.
- **Coordiante curve**: intersection of two coordinate surfaces.
- **Orthogonal curvilinear system**: where tangent vectors $\underline{e}_u, \underline{e}_v, \underline{e}_w$ are mutually orthogonal at any point P. Orientation of these vectors may depend on P.
- Let g invertible map from u-space to x-space, $g(u) = x$. Distortion factor $g'(u)$ is **Jacobian** of g. $\mathrm{d}x = g'(u)\,\mathrm{d}u$ so **method of substitution** is

$$\int_a^b f(x)\,\mathrm{d}x = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u)\,\mathrm{d}u$$

- In two dimensions, **Jacobian** for maps $\left(\tilde{g}, \tilde{h}\right)$ is

$$J\left(\tilde{g}, \tilde{h}\right) = \begin{vmatrix} \partial_x \tilde{g} & \partial_y \tilde{g} \\ \partial_x \tilde{h} & \partial_y \tilde{h} \end{vmatrix} =: \frac{\partial(\tilde{g}, \tilde{g})}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(x, y)}$$

Distortion factor is $\left|J\left(\tilde{g}, \tilde{h}\right)\right|$. So

$$\mathrm{d}A_{uv} = \left|J\left(\tilde{g}, \tilde{h}\right)\right| \mathrm{d}A_{xy}$$

where $\mathrm{d}A_{xy} = \mathrm{d}x\,\mathrm{d}y$ and

$$\mathrm{d}A_{xy} = |J(g, h)|\,\mathrm{d}A_{uv} = \left|J\left(\tilde{g}, \tilde{h}\right)\right|^{-1} \mathrm{d}A_{uv}$$

So

$$\iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{R'} f(g(u, v), h(u, v))|J(g, h)|\,\mathrm{d}u\,\mathrm{d}v$$

where R mapped to R' by $\left(\tilde{g}, \tilde{h}\right)$.
- $\mathrm{d}A_{uv}$ is parallelogram-shaped.
- In three dimensions,

$$\iiint_R f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{R'} f(g(u, v, w), h(u, v, w), k(u, v, w))\left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$$

where $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| := J(g, h, k)$.
- $\mathrm{d}V_{uvw}$ is parallelopiped-shaped.
- **Gradient in Cartesian coordinates**: $\underline{\nabla} = \underline{e}_1\partial_x + \underline{e}_2\partial_y + \underline{e}_3\partial_z$.
- **Laplacian in Cartesian coordinates**: $\underline{\nabla}^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$.

- For 2D polar coordinates, let $\underline{r} = r\cos(\theta)\underline{e}_1 + r\sin(\theta)\underline{e}_2$, then

$$\begin{aligned}\tilde{\underline{e}}_r &:= \partial_r \underline{r} = \cos(\theta)\underline{e}_1 + \sin(\theta)\underline{e}_2, \\ \tilde{\underline{e}}_\theta &:= \partial_\theta \underline{r} = -r\sin(\theta)\underline{e}_1 + r\cos(\theta)\underline{e}_2\end{aligned}$$

- Let $x = g(u, v)$, $y = h(u, v)$, then **scale factors for mapping given by g and h** are $h_u := \|\partial_u \underline{r}\|$, $h_v := \|\partial_v \underline{r}\|$.
- Unit vectors corresponding to $\tilde{\underline{e}}_r$ and $\tilde{\underline{e}}_\theta$ are $\underline{e}_r = \tilde{\underline{e}}_r$ and $\underline{e}_\theta = \frac{1}{r}\tilde{\underline{e}}_\theta$ which form orthonormal basis.
- $\mathrm{d}\underline{r} = \partial_r \underline{r}\,\mathrm{d}r + \partial_\theta \underline{r}\,\mathrm{d}\theta = \mathrm{d}r\underline{e}_r + r\,\mathrm{d}\theta\underline{e}_\theta$ by chain rule.
- **Gradient in polar coordinates**: $\underline{\nabla} = \underline{e}_r \partial_r + \underline{e}_\theta \frac{1}{r}\partial_\theta$, obtained by comparing $\mathrm{d}f := \underline{\nabla} f \cdot \mathrm{d}\underline{r} = \partial_r f\,\mathrm{d}r + \partial_\theta f\,\mathrm{d}\theta$ for function $f(r, \theta)$.
- **Divergence in polar coordinates**: for $\underline{A}(r, \theta) = A_r \underline{e}_r + A_\theta \underline{e}_\theta$,

$$\underline{\nabla} \cdot \underline{A} = \frac{1}{r}(\partial_r(rA_r) + \partial_\theta A_\theta)$$

- **Laplacian in polar coordinates**: $\underline{\nabla}^2 = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$
- **Spherical polar coordinates**: $x = r\sin(\theta)\cos(\varphi)$, $y = r\sin(\theta)\sin(\varphi)$, $z = r\cos(\theta)$, $r \geq 0$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$.
- **Cylindrical polar coordinates**: $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$, $r \geq 0$, $\theta \in [0, 2\pi)$, $z \in \mathbb{R}$.
- **General formula for curl of vector in Cartesian coordinates**: for $\underline{A}(r, \theta, \varphi) = A_r \underline{e}_r + A_\theta \underline{e}_\theta + A_\varphi \underline{e}_\varphi$,

$$\underline{\nabla} \times \underline{A} = h_r^{-1} h_\theta^{-1} h_\varphi^{-1} \begin{vmatrix} h_r \underline{e}_r & h_\theta \underline{e}_\theta & h_\varphi \underline{e}_\varphi \\ \partial_r & \partial_\theta & \partial_\varphi \\ A_r h_r & A_\theta h_\theta & A_\varphi h_\varphi \end{vmatrix}$$

10. Generalised functions (distributions)

- **Unit step function (Heaviside)**:

$$\Theta(t - t_0) := \begin{cases} 0 \text{ if } t \leq t_0 \\ 1 \text{ if } t > t_0 \end{cases}$$

- Let $\Omega \subseteq \mathbb{R}^n$ open. $\psi : \Omega \to \mathbb{C}$ is **test function** if:
  - ψ is **smooth**: $\psi \in C^\infty(\Omega)$.
  - **Support** of ψ,

    $$\operatorname{supp}(\psi) := \overline{\{\underline{x} \in \Omega : \psi(\underline{x}) \neq 0\}}$$

    is compact (in this case, bounded).
- Space of test functions on Ω, $\mathcal{D}(\Omega)$, is vector space.
- Let $\psi \in \mathcal{D}(\mathbb{R}^n), \underline{\xi} \in \mathbb{R}^n, a \in \mathbb{R} - \{0\}, g \in C^\infty(\mathbb{R}^n)$. Then
  - $\psi(\underline{x} + \underline{\xi}), \psi(-\underline{x}), \psi(a\underline{x}) \in \mathcal{D}(\mathbb{R}^n)$.
  - $g(\underline{x})\psi(\underline{x}) \in \mathcal{D}(\mathbb{R}^n)$.
- Let $\Omega \subseteq \mathbb{R}^n$ open, then $\{\psi_m\}_{m\in\mathbb{N}} \subseteq \mathcal{D}(\Omega)$ **converges** to $\psi \in \mathcal{D}(\Omega)$ if:

- Exists compact $K \subseteq \Omega$ such that $\text{supp}(\psi), \text{supp}(\psi_m) \subseteq K$ for every $m \in \mathbb{N}$ and
  - $\{\psi_m\}$ converges **uniformly** to ψ in $\mathcal{D}(\Omega)$ and
  - sequence

$$D^k \psi_m := \psi_m^{(k)} := \frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_n}}{\partial x_1^{k_n}}$$

  converges uniformly to $D^k \psi$ for every multi-index $k = (k_1, ..., k_n)$, $k_i \in \mathbb{N}_0$, $|k| = k_1 + \cdots + k_n$. (Write $\psi_m \to_{\mathcal{D}} \psi$.)
- $\{\psi_m\}$ converges to ψ if:
  - Exists compact $K \subseteq \Omega$ such that $\text{supp}(\psi_i) \subseteq K$ for every i and
  - For every multi-index $k = (k_1, ..., k_n)$ and $|k| = k_1 + \cdots + k_n$ (including $|k| = 0$), $\|D^k \psi_m - D^k \psi\|_\infty \to 0$ as $m \to \infty$ where $\|f\|_\infty := \sup\{|f(\underline{x})| : \underline{x} \in \mathbb{R}^n\}$.
- Let $\Omega \subseteq \mathbb{R}^n$ open. **Distribution** is continuous linear map $T : \mathcal{D}(\Omega) \to \mathbb{R}$.
  - T **linear**: $T[a\psi + b\varphi] = aT[\psi] + bT[\varphi]$.
  - T **continuous**:
    $\forall \psi \in \mathcal{D}(\Omega), \forall \{\psi_m\} \subseteq D(\Omega), \psi_m \to_{\mathcal{D}} \psi \Longrightarrow T[\psi_m] \to T[\psi]$ as $m \to \infty$
- Space of distibutions with test functions in $\mathcal{D}(\Omega)$, written $\mathcal{D}'(\Omega)$, is vector space.
- **Dirac delta function** $\delta : \mathcal{D}(\mathbb{R}^n) \to \mathbb{R}$, is distribution

$$\delta[\psi] := \psi(\underline{0})$$

- Let $f \in C^0(\mathbb{R}^n)$. Then

$$T_f[\psi] := \int_{\mathbb{R}^n} f(\underline{x})\psi(\underline{x}) \, \mathrm{d}\underline{x}$$

  is a distribution.
- $f : \mathbb{R}^n \to \mathbb{R}$ **locally integrable** if for every compact $K \subseteq \mathbb{R}^n$,

$$\int_K f(\underline{x}) \, \mathrm{d}\underline{x} < \infty$$

- $L^1_{\text{loc}}(\mathbb{R}^n)$ is set of locally integrable functions on $\mathbb{R}^n$.
- $T \in \mathcal{D}'(\mathbb{R}^n)$ is **regular distribution** if for some $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $T[\psi] = T_f[\psi]$ for $\psi \in \mathcal{D}(\mathbb{R}^n)$.
- **Any two locally integrable functions that differ by finite amounts at isolated points define the same regular distribution**.
- Distribution T is **singular** if no $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that $T = T_f$.
- **Symbolically, in the sense of distributions**, can write singular distribution as

$$T[\psi] := \int_{\mathbb{R}^n} T[\underline{x}]\psi(\underline{x}) \, \mathrm{d}\underline{x} =: \langle T, \psi \rangle$$

  Note $T[\underline{x}]$ not a function.
- δ is singular distribution.
- **Sifting property** of δ:

$$\int_{\mathbb{R}^n} \delta(\underline{x})\psi(\underline{x})\,\mathrm{d}\underline{x} = \begin{cases} \psi(\underline{0}) & \text{if } \underline{x} = \underline{0} \\ 0 & \text{otherwise} \end{cases}$$

- **General sifting property** of δ:

$$\int_{\Omega} \delta(\underline{x})\psi(\underline{x}) = \begin{cases} \psi(\underline{0}) & \text{if } \underline{0} \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

- **Notation**: if $n = 1$, write $\delta(\underline{x}) = \delta(x)$, if $n = 2$, write $\delta(x)\delta(y)$, etc.
- **Distribution operation rules**:
  - **Addition**: $(T_1 + T_2)[\psi] = T_1[\psi] + T_2[\psi]$.
  - **Multiplication by constant**: $(cT)[\psi] = cT[\psi]$.
  - **Shifting of distribution** by $\underline{\xi} \in \mathbb{R}^n$:
    $T_{\underline{\xi}}[\psi(\underline{x})] := \int_{\mathbb{R}^n} T(\underline{x} - \underline{\xi})\psi(\underline{x})\,\mathrm{d}\underline{x} = \int_{\mathbb{R}^n} T(\underline{y})\psi(\underline{y} + \underline{\xi})\,\mathrm{d}\underline{y} =: T\left[\psi(\underline{x} + \underline{\xi})\right]$
  - **Transposition**:
    $T^t(\psi(\underline{x})) := \int_{\mathbb{R}^n} T(-\underline{x})\psi(\underline{x})\,\mathrm{d}\underline{x} = \int_{\mathbb{R}^n} T(\underline{y})\psi(-\underline{y})\,\mathrm{d}\underline{y} =: T[\psi(-\underline{x})]$
  - **Dilation** by $\alpha \in \mathbb{R}$:
    $T_{(\alpha)}[\psi(\underline{x})] := \int_{\mathbb{R}^n} T(\alpha\underline{x})\psi(\underline{x})\,\mathrm{d}\underline{x} = \frac{1}{|a|^n}\int_{\mathbb{R}^n} T(\underline{y})\psi\left(\frac{\underline{y}}{\alpha}\right)\mathrm{d}\underline{y} =: \frac{1}{|a|^n}T\left[\psi\left(\frac{\underline{x}}{\alpha}\right)\right]$
  - **Multiplication by smooth function φ**:

$$(\varphi T)[\psi] := T[\varphi\psi]$$

- Delta distribution **sifting property**:

$$\delta_\alpha[\psi] := \int_{\Omega} \delta(x - a)\psi(x)\,\mathrm{d}x = \begin{cases} \psi(a) & \text{if } a \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

- In sense of distributions, $\varphi(x)\delta(x - \xi) = \varphi(\xi)\delta(x - \xi)$.
- Symbolically, $\delta(\alpha x) = \frac{1}{|a|}\delta(x)$.
- If $f \in C^1(\Omega)$ has simple (multiplicity one) zeros at $\{x_1, ..., x_n\}$ then

$$\int_{\Omega} \delta(f(x))\psi(x)\,\mathrm{d}x = \sum_{i=1}^{n} \frac{\psi(x_i)}{|f'(x_i)|}$$

- Distributions T_1 and T_2 **equal** if

$$\forall\psi \in \mathcal{D}(\Omega), \int_{\Omega} T_1(x)\psi(x)\,\mathrm{d}x = \int_{\Omega} T_2(x)\psi(x)\,\mathrm{d}x$$

- nth **derivative** of distribution T:

$$T^{(n)}[\psi] = (-1)^n T\left[\psi^{(n)}\right]$$

- **In the sense of distributions**, $\Theta'(t) = \delta(t)$.
- **Leibniz rule**:

$$(\varphi T)' = \varphi' T + \varphi T', \quad (\varphi T)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \varphi^{(k)} T^{(n-k)}$$

- f **piecewise continuous** on (a, b) if (a, b) can be divided into finite number of sub-intervals such that:
  - f continuous on interior of each sub-interval and

- f tends to finite limit on boundary of each sub-interval as approached from interior of that sub-interval.
- f **piecewise smooth** if piecewise continuous and has piecewise continuous first derivatives.
- To calculate **derivative in sense of distributions** of **piecewise-smooth** f, with branches $f_1, ..., f_n$ and discontinuities at $x_1, ..., x_{k-1}$:
  - Let $\breve{f}(x) = f_1(x) + (f_2(x) - f_1(x))\Theta(x - x_1) + \cdots + (f_k - f_{k-1})\Theta(x - x_{k-1})$
  - Then differentiate $\breve{f}$.
- If Jacobian J of changes of variables from $\underline{x}$ to $\underline{\xi}$, then

$$\delta\left(\underline{x} - \underline{x_0}\right) = \frac{1}{|J|}\delta\left(\underline{\xi} - \underline{\xi_0}\right)$$

11. Sturm-Liouville Theory

- Let $f : [a, b] \to \mathbb{R}$, $a = x_0 < x_1 < \cdots < x_n = b$, $x_i^* \in [x_{i-1}, x_i]$ Let $\Delta = \sup_{1 \le i \le n}(x_i - x_{i-1})$, $\mathcal{R}(f) = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$. f **Riemann integrable** if exists real number, written $\int_a^b f(x)\,\mathrm{d}x$ such that

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\Delta \to 0} \mathcal{R}(f)$$

- **Lebesgue integration**: choose $y_0 \le \min(f)$, $y_n \ge \max(f)$, $y_0 < y_1 < \cdots < y_n$. Let

$$s_n := \sum_{i=1}^{n} y_{i-1} \cdot \mu\{x : y_{i-1} \le f(x) < y_i\}$$

  where $\mu\{x : y_{i-1} \le f(x) < y_i\}$ is measure of set $\{x : y_{i-1} \le f(x) < y_i\}$, i.e. sum of lengths of subintervals $[a, b]$ where $y_{i-1} \le f(x) \le y_i$. **Lebesgue integral** is limit of s_n as $n \to \infty$.
- **Riemann-Lebesgue theorem**: let $f : [a, b] \to \mathbb{R}$ bounded. Then f Riemann integrable iff f continuous everywhere except on set of measure zero (continuous almost everywhere).
- Measure of set with countable number of elements is zero.
- Measure of $[a, b]$: $\mu([a, b]) = b - a$. Also, $\mu([a, b] \times [c, d]) = (b - a)(d - c)$.
- If function Riemann integrable, then it is Lebesgue integrable.
- L^1: space of Lebesgue measurable and absolutely integrable functions.
- L^2: space of Lebesgue measurable functions with absolutely integrable squares.
- **Hilbert space** $\mathbb{H}$: real/complex vector space which:
  - has Hermitian inner product $\langle \cdot, \cdot \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$, with:
    - **Hermiticity**: $\langle \underline{u}, \underline{v} \rangle = \overline{\langle \underline{v}, \underline{u} \rangle}$.
    - **Anti-linearity in first entry**: $\langle a(\underline{u} + \underline{v}), \underline{w} \rangle = \overline{a}\langle \underline{u}, \underline{w} \rangle + \overline{a}\langle \underline{v}, \underline{w} \rangle$, $a \in \mathbb{C}$.
    - **Positivity**: $\langle \underline{u}, \underline{u} \rangle \ge 0$ and $\langle \underline{u}, \underline{u} \rangle = 0 \Longleftrightarrow \underline{u} = 0$.
  - is complete for inner product-induced norm:

$$\|\cdot\| : \mathbb{H} \to \mathbb{R}_{\ge 0}, \quad \|\underline{u}\| = (\langle \underline{u}, \underline{u} \rangle)^{1/2}$$

, with:

  - **Separation of points**: $\|\underline{u}\| = 0 \iff u = 0$.
  - **Absolute homogeneity**: $\|a\underline{u}\| = |a|\|\underline{u}\|$, $a \in \mathbb{C}$.
  - **Triangle inequality**: $\|\underline{u} + \underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$.
- Complex inner product **sesquilinear** as anti-linear in first entry, but linear in second.
- **Inner product space**: vector space with inner product and induced norm.
- **Metric** on vector space V: function $d : V \times V \to \mathbb{R}$, with:
  - $d(\underline{u}, \underline{v}) \geq 0$.
  - $d(\underline{u}, \underline{v}) = 0 \iff \underline{u} = \underline{v}$.
  - $d(\underline{u}, \underline{v}) = d(\underline{v}, \underline{u})$.
  - $d(\underline{u}, \underline{v}) + d(\underline{v}, \underline{w}) \geq d(\underline{u}, \underline{w})$.
- **Metric space**: pair (V, d).
- One metric given by $d(\underline{u}, \underline{v}) = \|\underline{u} - \underline{v}\|$. Sequence $\left\{\underline{v}_n\right\} \subseteq V$ **converges to $\underline{v} \in V$ in the mean (in norm)** if

$$\lim_{n\to\infty}\left\|\underline{v}_n - \underline{v}\right\| = 0 \iff \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \quad \left\|\underline{v}_n - \underline{v}\right\| < \varepsilon$$

- $\left\{\underline{v}_n\right\}$ **Cauchy sequence** if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m, n \geq N, \quad d\left(\underline{v}_n, \underline{v}_m\right) < \varepsilon$$

- Metric space **complete** if every Cauchy sequence in (V, d) converges in V.
- Let space V be function $[a, b] \to \mathbb{C}$. Let **weight** function $w : [a, b] \to \mathbb{R}_{\geq 0}$ with finitely many zeros. **Inner product with weight w**:

$$\langle \underline{u}, \underline{v} \rangle_w := \int_a^b \overline{u}(x) v(x) w(x) \,\mathrm{d}x$$

  Write $\langle \underline{u}, \underline{v} \rangle_{w=1}$ as $\langle \underline{u}, \underline{v} \rangle$.
- $W \subseteq V$ **dense in V** if

$$\forall v \in V, \forall \varepsilon > 0, \exists w \in W, \quad d(v, w) < \varepsilon$$

- **Linear Operator**: $(L, D(L))$, $D(L)$ is dense linear subspace of $\mathbb{H}$, $L : D(L) \to \mathbb{H}$ linear:

$$L(au + vb) = aL(u) + bL(v)$$

  L is the **operator**, $D(L)$ is **domain** of L.
- Linear operator $L : \mathbb{H}_1 \to \mathbb{H}_2$ **bounded** if for some $M \geq 0$,

$$\forall v \in \mathbb{H}_1, \quad \|Lv\|_{\mathbb{H}_2} \leq M\|v\|_{\mathbb{H}_1}$$

  If M doesn't exist, L **unbounded**.
- **Norm** of L is

$$\|L\| := \inf\left\{M : \forall h \in \mathbb{H}_1, \|Lv\|_{\mathbb{H}_2} \leq M\|v\|_{\mathbb{H}_1}\right\}$$

- To show L unbounded, find sequence $\{x_n\} \subset D(L)$ with $\|x_n\|_{\mathbb{H}_1} \leq M$ for some M, but $\|Lx_n\|_{\mathbb{H}_2} \to \infty$ as $n \to \infty$.
- **Adjoint** of $(L, D(L))$ is $(L^*, D(L^*))$ where $L* : D(L*) \to \mathbb{H}_1$,

$$\langle Lv_1, v_2 \rangle^{\mathbb{H}_2} = \langle v_1, L^* v_2 \rangle^{\mathbb{H}_1}, \quad v_1 \in D(L), v_2 \in D(L^*)$$

 and

$$D(L^*) := \left\{ v_2 \in \mathbb{H}_2 : \exists v_2^* \in \mathbb{H}_1, \forall v_1 \in D(L), \langle Lv_1, v_2 \rangle^{\mathbb{H}_2} = \langle v_1, v_2^* \rangle^{\mathbb{H}_1} \right\}$$

 For each $v_2 \in D(L*)$, $v_2^* = L^* v_2$ is unique.
- **Boundary value problem (BVP) on $[\boldsymbol{a}, \boldsymbol{b}]$**:

$$Lu(x) = f(x), a < x < b, \quad B_1(u) = B_2(u) = 0$$

- **Dirichlet boundary conditions**: $B_1(u) = u(a) = 0, B_2(u) = u(b) = 0$.
- **Neumann boundary conditions**: $B_1(u) = u'(a) = 0, B_2(u) = u'(b) = 0$.
- **Periodic boundary conditions**:
 $B_1(u) = u(a) - u(b) = 0, B_2(u) = u'(a) - u'(b) = 0$.
- **Mixed boundary conditions**:
 $B_1(u) = \alpha_1 u(a) + \beta_1 u'(a) = 0, B_2(u) = \eta_2 u(b) + \kappa_2 u'(b) = 0$
- **Initial value problem (IVP) on $[\boldsymbol{a}, \boldsymbol{b}]$**:

$$Lu(x) = f(x), a < x < b, \quad u(a) = 0, u'(a) = 0$$

- **Formal adjoint** of $L = p_0(x) d_x^2 + p_1(x) d_x + p_2(x)$ is

$$L^* := \overline{p_0} d_x^2 + (2\overline{p_0}' - \overline{p_1}) d_x + \overline{p_0}'' - \overline{p_1}' + \overline{p_2}$$

- Domain of $L = p_0(x) d_x^2 + p_1(x) d_x + p_2(x)$ is

$$D(L) := \left\{ u \in C^2([a, b]) : B_1(u) = B_2(u) = 0 \right\}$$

- **Green's formula**: let $L = p_0(x) d_x^2 + p_1(x) d_x + p_2(x)$, L^* be formal adjoint. Then

$$\langle Lu, v \rangle - \langle u, L^* v \rangle = \left[\overline{p_0}(v\overline{u}' - v'\overline{u}) + (\overline{p_1} - \overline{p_0}') v\overline{u} \right]_a^b$$

- For $L = p_0(x) d_x^2 + p_1(x) d_x + p_2(x)$, $D(L^*)$ consists of all functions v satisfying **adjoint boundary conditions**:

$$\left[\overline{p_0}(v\overline{u}' - v'\overline{u}) + (\overline{p_1} - \overline{p_0}') v\overline{u} \right]_a^b = 0 \quad \forall u \in C^2([a, b]) \text{ with } B_1(u) = B_2(u) = 0$$

- $(L, D(L))$ self-adjoint if $\langle Lu, v \rangle = \langle u, L^* v \rangle$ (boundary terms vanish).
- BVP $Lu(x) = f(x)$, $B_1(u) = B_2(u) = 0$ **self-adjoint** if $L = L^*$ and $D(L) = D(L^*)$ (so adjoint boundary conditions equal original boundary conditions) ($(L, D(L))$ is self-adjoint).
- If $L = p_0(x) d_x^2 + p_1(x) d_x + p_2(x)$ with real-valued coefficients and $p_1 = {p_0}'$, then

$$L^* = d_x(p_0 d_x) + p_2 = L$$

 L is **formally self-adjoint** with respect to inner product. L is **Sturm-Liouville operator**. L Sturm-Liouville iff ${p_0}' = p_1$.
- Let $L = p_0(x) d_x^2 + p_1(x) d_x + p_2(x)$, then

$$\mathfrak{L} := \rho L = d_x(\rho p_0 d_x) + \rho p_2, \quad \rho = \frac{1}{p_0} \exp\left(\int \frac{p_1}{p_0} \,\mathrm{d}x\right)$$

is Sturm-Liouville.

- **Eigenfunction** u_n with **eigenvalue** λ_n with respect to weight function $w(x)$ satisfies $Lu_n(x) = \lambda_n w(x) u_n(x)$.
- **Method of separation of variables**: write $U(x,t) = T(t)u(x)$ when solving PDE.
- $[a,b]$ **natural interval** if $p_0(a) = p_0(b) = 0$ and $p_0(x) > 0$ for $x \in (a,b)$.
- **Sturm-Liouville eigenvalue problem**: $\mathfrak{L}u_n(x) + \lambda_n w(x) u_n(x) = 0$.
- For Sturm-Liouville eigenvalue problem:
  - Eigenvalues real.
  - Eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to inner product $\langle u, v \rangle_w$.
  - Eigenfunctions can be chosen to be real.
  - Eigenvalues of regular Sturm-Liouville eigenfunction problem $(|\alpha_1| + |\beta_1| > 0, |\eta_2| + |\kappa_2| > 0)$ are simple (multiplicity one).
  - Set of eigenvalues is countably infinite and monotonically increasing sequence: $\lambda_1 < \lambda_2 < \cdots$ and $\lim_{n\to\infty} \lambda_n = \infty$.
  - For regular SL problem, can write **generalised Fourier expansion (eigenfunction expansion)** of u as

$$u = \sum_{n=0}^{\infty} \langle \hat{u}_n, u \rangle_w \hat{u}_n$$

    for normalised eigenfunctions $\hat{u}_n$.
  - If $\mathfrak{L}u_n(x) + \lambda_n w(x) u_n(x) = f(x)$, then $f(x) = \sum_{n=0}^{\infty} \langle \hat{u}_n, f \rangle_w \hat{u}_n$. Equate eigenfunction of f with eigenfunction expansion of u in $\mathfrak{L}u_n(x) + \lambda_n w(x) u_n(x)$ and take inner product with $\hat{u}_m$ to determine $c_m = \langle \hat{u}_m, u \rangle$
- If f piecewise smooth on $[a,b]$, for all $x \in (a,b)$,

$$\frac{1}{2}(f(x_+) + f(x_-)) := \frac{1}{2}\left(\lim_{\varepsilon\to 0} f(x+\varepsilon) + \lim_{\varepsilon\to 0} f(x-\varepsilon)\right) = \sum_{n=0}^{\infty} \langle \hat{u}_n, u \rangle_w \hat{u}_n(x)$$

- **Completeness of eigenfunctions**:

$$\sum_{n=0}^{\infty} \hat{u}_n(y)\hat{u}_n(x)w(y) = \delta(x-y) = \delta(y-x) = \sum_{n=0}^{\infty} \hat{u}_n(x)\hat{u}_n(y)w(x)$$

12. Green's functions

- **IN/IN IVP**: $Lu(t) = f(t)$, $u(a) = \gamma_1 \neq 0$, $u'(a) = \gamma_2 \neq 0$.
- **IN/HOM IVP**: $Lu(t) = f(t)$, $u(a) = u'(a) = 0$.
- **HOM/IN IVP**: $Lu(t) = 0$, $u(a) = \gamma_1 \neq 0$, $u'(a) = \gamma_2 \neq 0$.
- Similarly for BVP.

- **BVP** boundary conditions: $B_1(u) = \alpha_1 u(a) + \beta_1 u'(a) + \eta_1 u(b) + \kappa_1 u'(b) = \gamma_1$, $B_2(u) = \alpha_2 u(a) + \beta_2 u'(a) + \eta_2 u(b) + \kappa_2 u'(b) = \gamma_2$. If $\gamma_1 = \gamma_2 = 0$, conditions are **homogeneous**. If $\eta_1 = \kappa_1 = \alpha_2 = \beta_2 = 0$, conditions are **separate**.
- u_1, u_2 **linearly independent** if $c_1 u_1(x) + c_2 u_2(x) = 0$ only satisfied by $c_1 = c_2 = 0$.
- **Wronskian** of u_1, u_2:

$$W(u_1, u_2) := \begin{vmatrix} u_1 & u_2 \\ u_1{}' & u_2{}' \end{vmatrix} = u_1 u_2{}' - u_1{}' u_2$$

- If $u_1, u_2 \in C^1([a, b])$ and $W(u_1, u_2)(x_0) \neq 0$ for some $x_0 \in [a, b]$ then u_1, u_2 linearly independent on $[a, b]$.
- If u_1, u_2 solutions of $Lu = 0$, Wronskian either identically zero or never zero on $[a, b]$. u_1, u_2 linearly dependent iff Wronskian identically zero.
- **To solve IN/IN IVP** $Lu(t) = f(t), u(0) = u_0, u'(0) = v_0$:
  - Solve HOM/IN IVP:

    $$Lu_{\text{hom}}(t) = 0, u_{\text{hom}}(0) = u_0, u_{\text{hom}}{}'(0) = v_0$$

    $u_{\text{hom}}(t) = c_1 \tilde{u}_1(t) + c_2 \tilde{u}_2(t)$ where $\tilde{u}_1, \tilde{u}_2$ linearly independent solutions.
  - Solve IN/HOM IVP:

    $$Lu_p(t) = f(t), u_p(0) = 0, u_p{}'(0) = 0$$

    .
  - General solution: $u(t) = u_{\text{hom}}(t) + u_p(t)$.
- **To solve IN/HOM IVP**:
  - Let $f(t) = \int_0^\infty \delta(t - \tau) f(\tau) \, \mathrm{d}\tau$.
  - $L_t G(t, \tau) = \delta(t - \tau)$, $G(0, \tau) = 0 = \partial_t G(0, \tau)$.
  - $G(0, \tau) = 0 = \partial_t G(0, \tau)$.
  - G continuous at $t = \tau$:

    $$\lim_{\varepsilon \to 0^+} G(\tau + \varepsilon, \tau) = \lim_{\varepsilon \to 0^+} G(\tau - \varepsilon, \tau)$$

  - Jump discontinuity of $\partial_t G$ at $t = \tau$:

    $$\lim_{\varepsilon \to 0^+} \left(\frac{\partial G}{\partial t}(\tau + \varepsilon, \tau) - \frac{\partial G}{\partial t}(\tau - \varepsilon, \tau) \right) = \frac{1}{p_0(\tau)}$$

  - Define ansatz:

    $$\begin{aligned} G(t, \tau) &= A_1(\tau) u_1(t) + B_1(\tau) u_2(t) \quad \text{for } t < \tau \\ G(t, \tau) &= A_2(\tau) u_1(t) + B_2(\tau) u_2(t) \quad \text{for } \tau < t \end{aligned}$$

    where u_1, u_2 linearly independent solutions of $Lu = 0$.
  - For $t < \tau$, $G(0, \tau) = 0 = \partial_t G(0, \tau)$ which should give $A_1(\tau) = B_1(\tau) = 0$ so $G(t, \tau) = 0$ for $t < \tau$.
  - For $t > \tau$, use jump discontinuity of $\partial_t G$ and continuity of G to find $A_2(\tau)$ and $B_2(\tau)$.

- $$G(t,\tau) = \begin{cases} 0 & \text{if } t < \tau \\ \frac{u_1(\tau)u_2(t) - u_2(\tau)u_1(t)}{p_0(\tau)W(u_1,u_2)(\tau)} & \text{if } t > \tau \end{cases}$$
- **Final solution**:

  $$u_p(t) = \int_0^\infty G(t,\tau)f(\tau)\,\mathrm{d}\tau$$

  $G(t,\tau)$ is **Green's function** encoding response of system at time t to unit impulse at time τ.

- **To solve IN/IN BVP** $Lu(x) = f(x)$, $u(a) = u_a, u(b) = u_b$:
  - Solve HOM/IN BVP:

    $$Lu_{\text{hom}}(x) = 0, u_{\text{hom}}(a) = u_a, u_{\text{hom}}(b) = u_b$$

    $u_{\text{hom}}(x) = c_1\tilde{u}_1(x) + c_2\tilde{u}_2(x)$ where $\tilde{u}_1, \tilde{u}_2$ linearly independent solutions.
  - Solve IN/HOM BVP:

    $$Lu_p(x) = f(x), u_p(a) = 0, u_p(b) = 0$$

  - General solution: $u(x) = u_{\text{hom}}(x) + u_p(x)$.
- **To solve IN/HOM BVP**:
  - $L_x G(x,\xi) = \delta(x-\xi)$, $G(a,\xi) = 0$ for $x < \xi$, $G(b,\xi) = 0$ for $\xi < x$.
  - Define ansatz:

    $$\begin{aligned} G(x,\xi) &= A_1(\xi)u_1(x) + B_1(\xi)u_2(x) \quad \text{for } x < \xi \\ G(x,\xi) &= A_2(\xi)u_1(x) + B_2(\xi)u_2(x) \quad \text{for } \xi < x \end{aligned}$$

    where u_1, u_2 linearly independent solutions of $Lu = 0$.
  - G continuous at $x = \xi$:

    $$\lim_{\varepsilon\to 0^+} G(\xi+\varepsilon,\xi) = \lim_{\varepsilon\to 0^+} G(\xi-\varepsilon,\xi)$$

  - Jump discontinuity of $\partial_x G$ at $x = \xi$:

    $$\lim_{\varepsilon\to 0^+}\left(\frac{\partial G}{\partial x}(\xi+\varepsilon,\xi) - \frac{\partial G}{\partial x}(\xi-\varepsilon,\xi)\right) = \frac{1}{p_0(\xi)}$$

  - Use continuity, jump discontinuity and boundary conditions to find A_1, B_1, A_2, B_2.
  - **Final solution**:

    $$u_p(x) = \int_a^b G(x,\xi)f(\xi)\,\mathrm{d}\xi$$

  - **Note**: if boundary conditions are $B_1(u) = \gamma_1, B_2(u) = \gamma_2$, G must satisfy $B_1(G) = 0, B_2(G) = 0$.
- **Laplace equation**: $\underline{\nabla}^2 u = u_{xx} + u_{yy} = 0$
- **Poisson's equation**: $\underline{\nabla}^2 u(x,y) = f(x,y)$.
- **Fundamental solution of Laplace's equation**:

$$G_2(\underline{x}) := \frac{1}{2\pi}\ln(r), \quad r = |\underline{x}|$$

- **Fundamental solution of Laplace's equation in plane with pole at $\boldsymbol{x} = \boldsymbol{\xi}$**:

$$G_2(\underline{x}, \underline{\xi}) := G_2(\underline{x} - \underline{\xi}) = \frac{1}{2\pi}\ln|\underline{x} - \underline{\xi}|$$

- **Green's first identity**: let $\underline{F} = \underline{\nabla}u$ in Divergence theorem:

$$\int_\Omega \underline{\nabla}^2 u \,\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega} \underline{\nabla}u \cdot \underline{n}\,\mathrm{d}s = \int_{\partial\Omega} \underline{n} \cdot \underline{\nabla}u\,\mathrm{d}s = \int_{\partial\Omega} \partial_n u\,\mathrm{d}s$$

- **Green's second identity**: let $F = u\underline{\nabla}v$ in Divergence theorem:

$$\int_\Omega \underline{\nabla} \cdot (u\underline{\nabla}v) = \int_\Omega \left(u\underline{\nabla}^2 v + \underline{\nabla}u \cdot \underline{\nabla}v\right)\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega} u\underline{\nabla}v \cdot \underline{n}\,\mathrm{d}s = \int_{\partial\Omega} u\partial_n v\,\mathrm{d}s$$

- **Green's third identity**: interchange u and v in second identity, subtract one from other:

$$\int_\Omega \left(u\underline{\nabla}^2 v - v\underline{\nabla}^2 u\right)\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega} (u\partial_n v - v\partial_n u)\,\mathrm{d}s$$

- **Dirichlet problem**: IN/IN BVP

$$\underline{\nabla}^2 u(\underline{x}) = f(\underline{x}), u(\underline{x})_{\partial\Omega} = g(\underline{x})$$

  To solve:

  - Subtract function G_{reg} from G_2 so that $G := G_2 - G_{\text{reg}}$ satisfies $\underline{\nabla}^2 G(\underline{x}, \underline{\xi}) = \delta(\underline{x} - \underline{\xi})$, $G(\underline{x}, \underline{\xi}) = 0 \Longleftrightarrow G_2 = G_{\text{reg}}$ for $x \in \partial\Omega$. G is **Green's function for Dirichlet problem on domain $\boldsymbol{\Omega}$**. G_{reg} must satisfy Laplace equation on Ω.
  - **Full solution**:

$$u(\underline{\xi}) = \int_\Omega G(\underline{x}, \underline{\xi}) f(\underline{x})\,\mathrm{d}x + \int_{\partial\Omega} g(\underline{x})\partial_n G(\underline{x}, \underline{\xi})\,\mathrm{d}s$$

    $\underline{n}$ is unit normal to Ω at $\underline{x}$ pointing outwards.

- To find G_{reg}, use **method of images**:
  - Fix point $P \in \Omega$ with position vector $\underline{\xi}_0$, let $Q \in \Omega$ have position vector $\underline{x}$. Then $G_2\left(\underline{x}, \underline{\xi}_0\right) = \frac{1}{2\pi}\ln|PQ|$.
  - Let $P_1, ..., P_n$ be reflection of P in boundary lines of $\partial\Omega$ (repeat reflection until back to P). Label $P_1, ..., P_n$ with alternating $-$ and $+$. Then

$$-G_{\text{reg}} = -\frac{1}{2\pi}\ln|QP_1| + \frac{1}{2\pi}\ln|QP_2| - \cdots - \ln|QP_n|$$

  - **Note**: if $\partial\Omega$ is circle radius R, OP_1 must satisfy $|OP| \cdot |OP_1| = R^2$ so

$$\tilde{\underline{\xi}}_0 := OP_1 = \frac{R^2}{\left|\underline{\xi}_0\right|^2}\underline{\xi}_0$$

- Check if G_{reg} satisfies $\underline{\nabla_x}^2 G_{\text{reg}}\left(\underline{x}, \underline{\xi}_0\right) = 0$ and $G_{\text{reg}} = G_2$ on $\partial\Omega$. If $G_{\text{reg}} \neq G_2$, add constant c to G_{reg} so that $G_{\text{reg}} = G_2$.