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# 1. The real numbers

## 1.1. Conventions on sets and functions

**Definition.** For  $f : X \rightarrow Y$ , **preimage** of  $Z \subseteq Y$  is

$$f^{-1}(Z) := \{x \in X : f(x) \in Z\}$$

**Definition.**  $f : X \rightarrow Y$  **injective** if

$$\forall y \in f(X), \exists! x \in X : y = f(x)$$

**Definition.**  $f : X \rightarrow Y$  **surjective** if  $Y = f(X)$ .

**Proposition.** Let  $f : X \rightarrow Y$ ,  $A, B \subseteq X$ , then

$$\begin{aligned} f(A \cap B) &\subseteq f(A) \cap f(B), \\ f(A \cup B) &= f(A) \cup f(B), \\ f(X) - f(A) &\subseteq f(X - A) \end{aligned}$$

**Proposition.** Let  $f : X \rightarrow Y$ ,  $C, D \subseteq Y$ , then

$$\begin{aligned} f^{-1}(C \cap D) &= f^{-1}(C) \cap f^{-1}(D), \\ f^{-1}(C \cup D) &= f^{-1}(C) \cup f^{-1}(D), \\ f^{-1}(Y - C) &= X - f^{-1}(C) \end{aligned}$$

## 1.2. The real numbers

**Definition.**  $a \in \mathbb{R}$  is an **upper bound** of  $E \subseteq \mathbb{R}$  if  $\forall x \in E, x \leq a$ .

**Definition.**  $c \in \mathbb{R}$  is a **least upper bound (supremum)** of  $E$ ,  $c = \sup(E)$ , if  $c \leq a$  for every upper bound  $a$ .

**Definition.**  $a \in \mathbb{R}$  is an **lower bound** of  $E \subseteq \mathbb{R}$  if  $\forall x \in E, x \geq a$ .

**Definition.**  $c \in \mathbb{R}$  is a **greatest lower bound (infimum)**,  $c = \inf(E)$ , if  $c \geq a$  for every lower bound  $a$ .

**Theorem** (Completeness axiom of the real numbers). Every  $E \subseteq \mathbb{R}$  with an upper bound has a least upper bound. Every  $E \subseteq \mathbb{R}$  with a lower bound has a greatest lower bound.

**Proposition** (Archimedes' principle).

$$\forall x \in \mathbb{R}, \exists n \in \mathbb{N} : n > x$$

**Remark.** Every non-empty subset of  $\mathbb{N}$  has a minimum.

**Proposition.**  $\mathbb{Q}$  is dense in  $\mathbb{R}$ :

$$\forall x < y \in \mathbb{R}, \exists r \in \mathbb{Q} : r \in (x, y)$$

## 1.3. Sequences, limits and series

**Definition.**  $l \in \mathbb{R}$  is **limit** of  $(x_n)$  ( $(x_n)$  converges to  $l$ ) if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |x_n - l| < \varepsilon$$

A sequence **converges in  $\mathbb{R}$  (is convergent)** if it has a limit  $l \in \mathbb{R}$ . Limit  $l = \lim_{n \rightarrow \infty} x_n$  is unique.

**Definition.**  $(x_n)$  **tends to infinity** if

$$\forall K > 0, \exists N \in \mathbb{N} : \forall n \geq N, \quad x_n > K$$

**Definition.** **Subsequence** of  $(x_n)$  is sequence  $(x_{n_j})$ ,  $n_1 < n_2 < \dots$ .

**Definition.** **Limit inferior** of sequence  $x_n$  is

$$\liminf_{n \rightarrow \infty} x_n := \sup_{n \in \mathbb{N}} \left\{ \inf_{m \geq n} x_m \right\} = \lim_{n \rightarrow \infty} \left( \inf_{m \geq n} x_m \right)$$

**Definition.** **Limit superior** of sequence  $x_n$  is

$$\limsup_{n \rightarrow \infty} x_n := \inf_{n \in \mathbb{N}} \left\{ \sup_{m \geq n} x_m \right\} = \lim_{n \rightarrow \infty} \left( \sup_{m \geq n} x_m \right)$$

**Proposition.** Let  $(x_n)$  bounded,  $l \in \mathbb{R}$ . Then  $l = \limsup x_n$  iff both of the following hold:

- $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n < l + \varepsilon$ .
- $\forall \varepsilon > 0, \forall N \in \mathbb{N} : \exists n \geq N : x_n > l - \varepsilon$ .

**Proposition.** Let  $(x_n)$  bounded,  $l \in \mathbb{R}$ . Then  $l = \liminf x_n$  iff both of the following hold:

- $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n > l - \varepsilon$ .
- $\forall \varepsilon > 0, \forall N \in \mathbb{N} : \exists n \geq N : x_n < l + \varepsilon$ .

**Theorem** (Bolzano-Weierstrass). Every bounded sequence has a convergent subsequence.

**Proposition.** Let  $(x_n)$  bounded. There exists convergent subsequence with limit  $\limsup x_n$  and convergent subsequence with limit  $\liminf x_n$ .

**Proposition.** Let  $(x_n)$  bounded, then  $(x_n)$  is convergent iff  $\limsup x_n = \liminf x_n$ .

**Theorem** (Monotone convergence theorem for sequences). Monotone sequence converges in  $\mathbb{R}$  or tends to either  $\infty$  or  $-\infty$ .

**Definition.**  $(x_n)$  is **Cauchy sequence** if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n, m \geq N, \quad |x_n - x_m| < \varepsilon$$

**Theorem.** Every Cauchy sequence in  $\mathbb{R}$  is convergent.

## 1.4. Open and closed sets

**Definition.**  $U \subseteq \mathbb{R}$  is **open** if

$$\forall x \in U, \exists \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \subseteq U$$

**Proposition.** Arbitrary unions of open sets are open. Finite intersections of open sets are open.

**Definition.**  $x \in \mathbb{R}$  is **point of closure (limit point)** for  $E \subseteq \mathbb{R}$  if

$$\forall \varepsilon > 0, \exists y \in E : |x - y| < \varepsilon$$

Equivalently,  $x$  is point of closure of  $E$  if every open interval containing  $x$  contains a point of  $E$ .

**Definition.** Closure of  $E$ ,  $\overline{E}$ , is set of points of closure. Note  $E \subseteq \overline{E}$ .

**Definition.**  $F$  is **closed** if  $F = \overline{F}$ .

**Proposition.**  $\overline{A \cup B} = \overline{A} \cup \overline{B}$ . If  $A \subset B \subseteq \mathbb{R}$  then  $\overline{A} \subset \overline{B}$ .

**Proposition.** For any set  $E$ ,  $\overline{E}$  is closed, i.e.  $\overline{\overline{E}} = \overline{E}$ .

**Proposition.**  $E \subseteq \mathbb{R}$  is closed iff  $\mathbb{R} - E$  is open.

**Proposition.** Arbitrary intersections of closed sets are closed. Finite unions of closed sets are closed.

**Definition.** Collection  $C$  of subsets of  $\mathbb{R}$  **covers** (is a **covering** of)  $F \subseteq \mathbb{R}$  if  $F \subseteq \bigcup_{S \in C} S$ . If each  $S$  in  $C$  open,  $C$  is **open covering**. If  $C$  is finite,  $C$  is **finite covering**.

**Definition.** Covering  $C$  of  $F$  **contains a finite subcover** if exists  $\{S_1, \dots, S_n\} \subseteq C$  with  $F \subseteq \bigcup_{i=1}^n S_i$  (i.e. a finite subset of  $C$  covers  $F$ ).

**Definition.**  $F$  is **compact** if any open covering of  $F$  contains a finite subcover.

**Example.**  $\mathbb{R}$  is not compact,  $[a, b]$  is compact.

**Theorem** (Heine Borel).  $F$  compact iff  $F$  closed and bounded.

## 1.5. Continuity, pointwise and uniform convergence of functions

**Definition.** Let  $E \subseteq \mathbb{R}$ .  $f : E \rightarrow \mathbb{R}$  is **continuous at**  $a \in E$  if

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in E, |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$$

$f$  is **continuous** if continuous at all  $y \in E$ .

**Definition.**  $\lim_{x \rightarrow a} f(x) = l$  if

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in E, |x - a| < \delta \implies |f(x) - l| < \varepsilon$$

**Proposition.**  $\lim_{x \rightarrow a} f(x) = l$  iff for every sequence  $(a_n)$  with  $\lim_{n \rightarrow \infty} a_n = a$ ,  $\lim_{n \rightarrow \infty} f(a_n) = l$ .

**Proposition.**  $f$  is continuous at  $a \in E$  iff  $\lim_{x \rightarrow a} f(x) = f(a)$  (and this limit exists).

**Definition.**  $f : E \rightarrow \mathbb{R}$  is **uniformly continuous** if

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x, y \in E, |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

**Proposition.** Let  $F$  closed and bounded,  $f : F \rightarrow \mathbb{R}$  continuous. Then  $f$  is uniformly continuous.

**Definition.** Let  $f_n : E \rightarrow \mathbb{R}$  sequence of functions,  $f : E \rightarrow \mathbb{R}$ .  $(f_n)$  **converges pointwise** to  $f$  if

$$\forall \varepsilon > 0, \forall x \in E, \exists N \in \mathbb{N} : \forall n \geq N, |f_n(x) - f(x)| < \varepsilon$$

$(f_n)$  **converges uniformly** to  $f$  is

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \forall x \in E, |f_n(x) - f(x)| < \varepsilon$$

**Theorem.** Let  $f_n : E \rightarrow \mathbb{R}$  sequence of continuous functions converging uniformly to  $f : E \rightarrow \mathbb{R}$ . Then  $f$  is continuous.

**Definition.**  $P = \{x_0, \dots, x_n\}$  is **partition** of  $[a, b]$  if  $a = x_0 < \dots < x_n = b$ .

**Definition.**  $f : [a, b] \rightarrow \mathbb{R}$  is **piecewise linear** if there exists partition  $P = \{x_0, \dots, x_n\}$  and  $m_i, c_i \in \mathbb{R}$  such that

$$\forall i \in [n], \forall x \in (x_{i-1}, x_i), \quad f(x) = m_i x + c_i$$

$f$  is continuous on  $[a, b] - P$ .

**Definition.**  $g : [a, b] \rightarrow \mathbb{R}$  is **step function** if there exists partition  $P = \{x_0, \dots, x_n\}$  and  $m_i \in \mathbb{R}$  such that

$$\forall i \in [n], \forall x \in (x_{i-1}, x_i), \quad g(x) = m_i$$

$g$  is continuous on  $[a, b] - P$ .

**Theorem.** Let  $f : E \rightarrow \mathbb{R}$  continuous,  $E$  closed and bounded. Then there exist continuous piecewise linear  $f_n$  with  $f_n \rightarrow f$  uniformly, and step functions  $g_n$  with  $g_n \rightarrow f$  uniformly.

**Definition.**  $f : E \rightarrow \mathbb{R}$  is **Lipschitz** if

$$\exists C > 0 : \forall x, y \in E, \quad |f(x) - f(y)| \leq C|x - y|$$

**Definition.**  $f : E \rightarrow \mathbb{R}$  is **bi-Lipschitz** if

$$\exists C > 0 : \forall x, y \in E, \quad C^{-1}|x - y| \leq |f(x) - f(y)| \leq C|x - y|$$

## 1.6. The extended real numbers

**Definition.** **Extended reals** are  $\mathbb{R} \cup \{-\infty, \infty\}$  with the order relation  $-\infty < \infty$  and  $\forall x \in \mathbb{R}, -\infty < x < \infty$ .  $\infty$  is an upper bound and  $-\infty$  is a lower bound for every  $x \in \mathbb{R}$ , so  $\sup(\mathbb{R}) = \infty$ ,  $\inf(\mathbb{R}) = -\infty$ ,  $\sup(\emptyset) = -\infty$ ,  $\inf(\emptyset) = \infty$ .

- Addition:  $\forall a \in \mathbb{R}, a + \infty = \infty \wedge a + (-\infty) = -\infty$ .  $\infty + \infty = \infty - (-\infty) = \infty$ .  
 $\infty - \infty$  is undefined.
- Multiplication:  $\forall a > 0, a \cdot \infty = \infty, \forall a < 0, a \cdot \infty = -\infty$ . Also  $\infty \cdot \infty = \infty$ .
- $\limsup$  and  $\liminf$  are defined as

$$\limsup x_n := \inf\{\sup\{x_k : k \geq n\} : n \in \mathbb{N}\}, \quad \liminf x_n := \sup\{\inf\{x_k : k \geq n\} : n \in \mathbb{N}\}$$

**Definition.** Extended real number  $l$  is **limit** of  $(x_n)$  if either

- $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |x_n - l| < \varepsilon$ . Then  $(x_n)$  **converges to**  $l$ . or
- $\forall \Delta > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n > \Delta$  (limit is  $\infty$ ) or
- $\forall \Delta > 0, \exists N \in \mathbb{N} : \forall n \geq N, x_n < -\Delta$  (limit is  $-\infty$ ).

$(x_n)$  **converges in the extended reals** if it has a limit in the extended reals.

## 2. Further analysis of subsets of $\mathbb{R}$

## 2.1. Countability and uncountability

**Definition.**  $A$  is **countable** if  $A = \emptyset$ ,  $A$  is finite or there is a bijection  $\varphi : \mathbb{N} \rightarrow A$  (in which case  $A$  is **countably infinite**). Otherwise  $A$  is **uncountable**. **Enumeration** is bijection to  $A$  from  $[n]$  or  $\mathbb{N}$ .

**Proposition.** If there is surjection from countable set to  $A$ , or injection from  $A$  to countable set, then  $A$  is countable.

**Proposition.** Any subset of  $\mathbb{N}$  is countable.

**Proposition.**  $\mathbb{Q}$  is countable.

**Proposition.** If  $(a_n)$  is a nonnegative sequence and  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$  is a bijection then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\varphi(n)}$$

**Proposition.** If  $(a_{n,k})$  is a nonnegative sequence and  $\varphi : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$  is a bijection then

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{n,k} = \sum_{n=1}^{\infty} a_{\varphi(n)}$$

**Definition.**  $f : X \rightarrow Y$  is **monotone** if  $x \geq y \Rightarrow f(x) \geq f(y)$  or  $x \leq y \Rightarrow f(x) \leq f(y)$ .

**Proposition.** Let  $f$  be monotone on  $(a, b)$ . Then it is discontinuous on a countable set.

**Lemma.** Set of sequences in  $\{0, 1\}$ ,  $\{(x_n)_{n \in \mathbb{N}} : \forall n \in \mathbb{N}, x_n \in \{0, 1\}\}$  is uncountable.

**Theorem.**  $\mathbb{R}$  is uncountable.

## 2.2. The structure theorem for open sets

**Definition.** Collection  $\{A_i : i \in I\}$  of sets is **(pairwise) disjoint** if  $n \neq m \Rightarrow A_n \cap A_m = \emptyset$ .

**Theorem** (Structure theorem for open sets). Let  $U \subseteq \mathbb{R}$  open. Then exists countable collection of disjoint open intervals  $\{I_n : n \in \mathbb{N}\}$  such that  $U = \bigcup_{n \in \mathbb{N}} I_n$ .

## 2.3. Accumulation points and perfect sets

**Definition.**  $x \in \mathbb{R}$  is **accumulation point** of  $E \subseteq \mathbb{R}$  if  $x$  is point of closure of  $E - \{x\}$ . Equivalently,  $x$  is a point of closure if

$$\forall \varepsilon > 0, \exists y \in E : y \neq x \wedge |x - y| < \varepsilon$$

Equivalently, there exists a sequence of distinct  $y_n \in E$  with  $y_n \rightarrow x$  as  $n \rightarrow \infty$ .

**Proposition.** Set of accumulation points of  $\mathbb{Q}$  is  $\mathbb{R}$ .

**Proposition.** Set of accumulation points  $E'$  of  $E$  is closed.

**Definition.**  $E \subseteq \mathbb{R}$  is **isolated** if

$$\forall x \in E, \exists \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \cap E = \{x\}$$

**Proposition.**  $E$  is isolated iff it has no accumulation points.

**Definition.** Bounded set  $E$  is **perfect** if it equals its set of accumulation points.

**Theorem.** Every non-empty perfect set is uncountable.

## 2.4. The middle-third Cantor set

**Proposition.** Let  $\{F_n : n \in \mathbb{N}\}$  be collection of non-empty nested closed sets (so  $F_{n+1} \subseteq F_n$ ), one of which is bounded. Then

$$\bigcap_{n \in \mathbb{N}} F_n \neq \emptyset$$

**Definition.** The **middle third Cantor set** is defined by:

- Define  $C_0 := [0, 1]$
- Given  $C_n = \bigcup_{i=1}^{2^n} [a_i, b_i]$ ,  $a_1 < b_1 < a_2 < \dots < a_{2^n} < b_{2^n}$ , with  $|b_i - a_i| = 3^{-n}$ , define

$$C_{n+1} := \bigcup_{i=1}^{2^{n+1}} [a_i, a_i + 3^{-(n+1)}] \cup [b_i - 3^{-(n+1)}, b_i]$$

which is a union of  $2^{n+1}$  disjoint intervals, with all differences in endpoints equalling  $3^{-(n+1)}$ .

- The **middle third Cantor set** is

$$C := \bigcap_{n \in \mathbb{N}_0} C_n$$

Observe that if  $a$  is an endpoint of an interval in  $C_n$ , it is contained in  $C$ .

**Proposition.** The middle third Cantor set is closed, non-empty and equal to its set of accumulation points. Hence it is perfect and so uncountable.

**Definition.** Let  $k \in \mathbb{N} - \{1\}$ ,  $x \in [0, 1)$ .  $0.a_1 a_2 \dots$ ,  $a_i \in \{0, \dots, k-1\}$ , is a  **$k$ -ary expansion** of  $x$  if

$$x = \sum_{i \in \mathbb{N}} \frac{a_i}{k^i}$$

**Remark.** The  $k$ -ary expansion may not be unique, but there is a countable set  $E \subseteq [0, 1)$  such that every  $x \in [0, 1) - E$  has a unique  $k$ -ary expansion.

**Remark.** For every  $x \in C$ , the ternary ( $k = 3$ ) expansion of  $x$  is unique and

$$x = \sum_{i \in \mathbb{N}} \frac{a_i}{3^i}, \quad a_i \in \{0, 2\}$$

Moreover, every choice of sequence  $(a_i)$ ,  $a_i \in \{0, 2\}$ , gives  $x = \sum_{i \in \mathbb{N}} \frac{a_i}{3^i} \in C$ .

**Definition.** **Cantor-Lebesgue function**,  $g : [0, 1] \rightarrow [0, 1]$ , is defined by

$$g(x) := \begin{cases} \sum_{i \in \mathbb{N}} \frac{a_i/2}{2^i} & \text{if } x = \sum_{i \in \mathbb{N}} \frac{a_i}{3^i}, a_i \in \{0, 2\} \\ \sup\{g(y) : y \in C, y \leq x\} & \text{if } x \notin C \end{cases}$$

$g$  is a surjection, monotone and continuous.

## 2.5. $G_\delta, F_\sigma$

**Definition.**  $E \subseteq \mathbb{R}$  is  $G_\delta$  if  $E = \bigcap_{n \in \mathbb{N}} U_n$  with  $U_n$  open.

**Definition.**  $E \subseteq \mathbb{R}$  is  $F_\sigma$  if  $E = \bigcup_{n \in \mathbb{N}} F_n$  with  $F_n$  closed.

**Lemma.** Set of points where  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous is  $G_\delta$ .

## 3. Construction of Lebesgue measure

### 3.1. Lebesgue outer measure

**Definition.** Let  $I$  non-empty interval with endpoints  $a = \inf(I) \in \{-\infty\} \cup \mathbb{R}$  and  $b = \sup(I) \in \mathbb{R} \cup \{\infty\}$ . The **length** of  $I$  is

$$\ell(I) := b - a$$

and set  $\ell(\emptyset) = 0$ .

**Definition.** Let  $A \subseteq \mathbb{R}$ . **Lebesgue outer measure** of  $A$  is infimum of all sums of lengths of intervals covering  $A$ :

$$\mu^*(A) := \inf \left\{ \sum_{k \in \mathbb{N}} \ell(I_k) : A \subseteq \bigcup_{k \in \mathbb{N}} I_k, I_k \text{ intervals} \right\}$$

It satisfies **monotonicity**:  $A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$ .

**Proposition.** Outer measure is **countably subadditive**:

$$\mu^* \left( \bigcup_{k \in \mathbb{N}} E_k \right) \leq \sum_{k \in \mathbb{N}} \mu^*(E_k)$$

This implies **finite subadditivity**:

$$\mu^* \left( \bigcup_{k=1}^n E_k \right) \leq \sum_{k=1}^n \mu^*(E_k)$$

**Lemma.** We have

$$\mu^*(A) = \inf \left\{ \sum_{k \in \mathbb{N}} \ell(I_k) : A \subset \bigcup_{k \in \mathbb{N}} I_k, I_k \neq \emptyset \text{ open intervals} \right\}$$

**Proposition.** Outer measure of interval is its length:  $\mu^*(I) = \ell(I)$ .

### 3.2. Measurable sets

**Notation.**  $E^c = \mathbb{R} - E$ .

**Proposition.** Let  $E = (a, \infty)$ . Then

$$\forall A \subseteq \mathbb{R}, \quad \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

**Definition.**  $E \subseteq \mathbb{R}$  is **Lebesgue measurable** if

$$\forall A \subseteq \mathbb{R}, \quad \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$



Collection of such sets is  $\mathcal{F}_{\mu^*}$ .

**Lemma** (Excision Property). Let  $E$  Lebesgue measurable set with finite measure and  $E \subseteq B$ , then

$$\mu^*(B - E) = \mu^*(B) - \mu^*(E)$$

**Proposition.** If  $E_1, \dots, E_n$  Lebesgue measurable then  $\cup_{k=1}^n E_k$  is Lebesgue measurable. If  $E_1, \dots, E_n$  disjoint then

$$\mu^* \left( A \cap \bigcup_{k=1}^n E_k \right) = \sum_{k=1}^n \mu^*(A \cap E_k)$$

for any  $A \subseteq \mathbb{R}$ . In particular, for  $A = \mathbb{R}$ ,

$$\mu^* \left( \bigcup_{k=1}^n E_k \right) = \sum_{k=1}^n \mu^*(E_k)$$

**Remark.** Not every set is Lebesgue measurable.

**Definition.** Collection of subsets of  $\mathbb{R}$  is an **algebra** if contains  $\emptyset$  and closed under taking complements and finite unions: if  $A, B \in \mathcal{A}$  then  $\mathbb{R} - A, A \cup B \in \mathcal{A}$ .

**Remark.** A union of a countable collection of Lebesgue measurable sets is also the union of a countable disjoint collection of Lebesgue measurable sets: if  $\{A_k\}_{k \in \mathbb{N}}$  is countable collection of Lebesgue measurable sets, then let  $A_1' := A_1$  and for  $k > 1$ , define

$$A_k' := A_k - \bigcup_{i=1}^{k-1} A_i$$

then  $\{A_k'\}_{k \in \mathbb{N}}$  is disjoint union of Lebesgue measurable sets and  $\cup_{k \in \mathbb{N}} A_k' = \cup_{k \in \mathbb{N}} A_k$ .

**Proposition.** If  $E$  is countable union of Lebesgue measurable sets, then  $E$  is Lebesgue measurable. Also, if  $\{E_k\}_{k \in \mathbb{N}}$  is countable disjoint collection of Lebesgue measurable sets then

$$\mu \left( \bigcup_{k \in \mathbb{N}} E_k \right) = \sum_{k \in \mathbb{N}} \mu(E_k)$$

### 3.3. Abstract definition of a measure

**Definition.** Let  $X \subseteq \mathbb{R}$ . Collection of subsets of  $\mathcal{F}$  of  $X$  is  **$\sigma$ -algebra** if

- $\emptyset \in \mathcal{F}$
- $E \in \mathcal{F} \implies E^c \in \mathcal{F}$
- If  $\forall k \in \mathbb{N}, E_k \in \mathcal{F}$  then  $\cup_{k \in \mathbb{N}} E_k \in \mathcal{F}$ .

**Example.**

- Trivial examples are  $\mathcal{F} = \{\emptyset, \mathbb{R}\}$  and  $\mathcal{F} = \mathcal{P}(\mathbb{R})$ .
- Countable intersections of  $\sigma$ -algebras are  $\sigma$ -algebras.

**Definition.** Let  $\mathcal{F}$   $\sigma$ -algebra of  $X$ .  $\nu : \mathcal{F} \rightarrow \mathbb{R} \cup \{\pm\infty\}$  is **measure** satisfying

- $\nu(\emptyset) = 0$

- $\forall E \in \mathcal{F}, \nu(E) \geq 0$
- **Countable additivity:** if  $E_1, E_2, \dots \in \mathcal{F}$  are disjoint then

$$\nu\left(\bigcup_{k \in \mathbb{N}} E_k\right) = \sum_{k \in \mathbb{N}} \nu(E_k)$$

Elements of  $\mathcal{F}$  are **measurable** (as they are the only sets on which the measure  $\nu$  is defined).

**Proposition.** If  $\nu$  is measure then it satisfies:

- **Monotonicity:**  $A \subseteq B \implies \nu(A) \leq \nu(B)$ .
- **Countable subadditivity:**  $\nu(\bigcup_{k \in \mathbb{N}} E_k) \leq \sum_{k \in \mathbb{N}} \nu(E_k)$ .
- **Excision:** if  $B$  has finite measure, then  $A \subseteq B \implies \nu(B - A) = \nu(B) - \nu(A)$ .

### 3.4. Lebesgue measure

**Lemma.**  $\mathcal{F}_{\mu^*}$  is  $\sigma$ -algebra and contains every interval.

**Theorem** (Carathéodory Extension). Restriction of the  $\mu^*$  to  $\mathcal{F}_{\mu^*}$  is a measure.

**Theorem** (Hahn extension theorem). There exists unique measure  $\mu$  defined on  $\mathcal{F}_{\mu^*}$  for which  $\mu(I) = \ell(I)$  for any interval  $I$ .

**Definition.** The measure  $\mu$  of  $\mu^*$  restricted to  $\mathcal{F}_{\mu^*}$  is the **Lebesgue measure**. It satisfies  $\mu(I) = \ell(I)$  for any interval  $I$  and is translation invariant.

### 3.5. Sets of measure 0

**Proposition.** Middle-third Cantor set is Lebesgue measurable and has Lebesgue measure 0.

**Proposition.** Any countable set is Lebesgue measurable and has Lebesgue measure 0.

**Proposition.** Any  $E$  with  $\mu^*(E) = 0$  is Lebesgue measurable and has  $\mu(E) = 0$ .

**Lemma.** Let  $E$  Lebesgue measurable set with  $\mu(E) = 0$ , then  $\forall E' \subseteq E$ ,  $E'$  is Lebesgue measurable.

### 3.6. Continuity of measure

**Definition.** Countable collection  $\{E_k\}_{k \in \mathbb{N}}$  is **ascending** if  $\forall k \in \mathbb{N}, E_k \subseteq E_{k+1}$  and **descending** if  $\forall k \in \mathbb{N}, E_{k+1} \subseteq E_k$ .

**Theorem.** Every measure  $m$  satisfies:

- If  $\{A_k\}_{k \in \mathbb{N}}$  is ascending collection of measurable sets, then

$$m\left(\bigcup_{k \in \mathbb{N}} A_k\right) = \lim_{k \rightarrow \infty} m(A_k)$$

- If  $\{B_k\}_{k \in \mathbb{N}}$  is descending collection of measurable sets and  $m(B_1) < \infty$ , then

$$m\left(\bigcap_{k \in \mathbb{N}} B_k\right) = \lim_{k \rightarrow \infty} m(B_k)$$

### 3.7. An approximation result for Lebesgue measure

**Definition.** Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R})$  is smallest  $\sigma$ -algebra containing all intervals: for any other  $\sigma$ -algebra  $\mathcal{F}$  containing all intervals,  $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{F}$ .

$$\mathcal{B}(\mathbb{R}) := \bigcap \{ \mathcal{F} : \mathcal{F} \text{ } \sigma \text{-algebra containing all intervals} \}$$

$E \in \mathcal{B}(\mathbb{R})$  is **Borel** or **Borel measurable**.

**Lemma.** All open subsets of  $\mathbb{R}$ , closed subsets of  $\mathbb{R}$ ,  $G_\delta$  sets and  $F_\sigma$  sets are Borel.

**Proposition.** The following are equivalent:

- $E$  is Lebesgue measurable
- $\forall \varepsilon > 0, \exists$  open  $G : E \subseteq G \wedge \mu^*(G - E) < \varepsilon$
- $\forall \varepsilon > 0, \exists$  closed  $F : F \subseteq E \wedge \mu^*(E - F) < \varepsilon$
- $\exists G \in G_\delta : E \subseteq G \wedge \mu^*(G - E) = 0$
- $\exists F \in F_\sigma : F \subseteq E \wedge \mu^*(E - F) = 0$

## 4. Measurable functions

### 4.1. Definition of a measurable function

**Proposition.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ .  $f$  continuous iff  $\forall$  open  $U \subseteq \mathbb{R}, f^{-1}(U) \subseteq \mathbb{R}$  is open.

**Lemma.** Let  $f : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$  with  $E$  Lebesgue measurable. The following are equivalent:

- $\forall c \in \mathbb{R}, \{x \in E : f(x) > c\}$  is Lebesgue measurable.
- $\forall c \in \mathbb{R}, \{x \in E : f(x) \geq c\}$  is Lebesgue measurable.
- $\forall c \in \mathbb{R}, \{x \in E : f(x) < c\}$  is Lebesgue measurable.
- $\forall c \in \mathbb{R}, \{x \in E : f(x) \leq c\}$  is Lebesgue measurable.

The same statement holds for Borel measurable sets.

**Definition.**  $f : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$  is **(Lebesgue) measurable** if it satisfies any of the above properties and if  $E$  is Lebesgue measurable.  $f$  being **Borel measurable** is defined similarly.

**Corollary.** If  $f$  is Lebesgue measurable then for every  $B \in \mathcal{B}(\mathbb{R}), f^{-1}(B)$  is measurable. In particular, if  $f$  is Lebesgue measurable, preimage of any interval is measurable.

**Definition.** **Indicator function** on set  $A, \mathbb{1}_A : \mathbb{R} \rightarrow \{0, 1\}$ , is

$$\mathbb{1}_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

**Definition.**  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is **simple (measurable) function** if  $\varphi$  is measurable function that has finite codomain.

### 4.2. Fundamental aspects of measurable functions

**Definition.** Let  $E \subseteq F \subseteq \mathbb{R}$ , let  $f : F \rightarrow \mathbb{R}$ . **Restriction**  $f_E$  is function with domain  $E$  and for which  $\forall x \in E, f_E(x) = f(x)$ .

**Definition.** Real-valued function which is increasing or decreasing is **monotone**.

**Definition.** Sequence  $(f_n)$  on domain  $E$  is increasing if  $f_n \leq f_{n+1}$  on  $E$  for all  $n \in \mathbb{N}$ .

**Example.** Continuous functions are measurable.

**Definition.** For  $f_1 : E \rightarrow \mathbb{R}, \dots, f_n : E \rightarrow \mathbb{R}$ , define

$$\max\{f_1, \dots, f_n\}(x) := \max\{f_1(x), \dots, f_n(x)\}$$

$\min\{f_1, \dots, f_n\}$  is defined similarly.

**Proposition.** For finite family  $\{f_k\}_{k=1}^n$  of measurable functions with common domain  $E$ ,  $\max\{f_1, \dots, f_n\}$  and  $\min\{f_1, \dots, f_n\}$  are measurable.

**Definition.** For  $f : E \rightarrow \mathbb{R}$ , functions  $|f|, f^+, f^-$  defined on  $E$  are

$$|f|(x) := \max\{f(x), -f(x)\}, \quad f^+(x) := \max\{f(x), 0\}, \quad f^-(x) := \max\{-f(x), 0\}$$

**Corollary.** If  $f$  measurable on  $E$ , so are  $|f|, f^+$  and  $f^-$ .

**Proposition.** Let  $f : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$ . For measurable  $D \subseteq E$ ,  $f$  measurable on  $E$  iff restrictions of  $f$  to  $D$  and  $E - D$  are measurable.

**Theorem.** Let  $f, g : E \rightarrow \mathbb{R}$  measurable.

- **Linearity:**  $\forall \alpha, \beta \in \mathbb{R}, \alpha f + \beta g$  is measurable.
- **Products:**  $fg$  is measurable.

**Proposition.** Let  $f_n : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$  be sequence of measurable functions that converges pointwise to  $f : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$ . Then  $f$  is measurable.

**Lemma** (Simple approximation lemma). Let  $f : E \rightarrow \mathbb{R}$  measurable and bounded, so  $\exists M \geq 0 : \forall x \in E, |f|(x) < M$ . Then  $\forall \varepsilon > 0$ , there exist simple measurable functions  $\varphi_\varepsilon, \psi_\varepsilon : E \rightarrow \mathbb{R}$  such that

$$\forall x \in E, \quad \varphi_\varepsilon(x) \leq f(x) \leq \psi_\varepsilon(x) \wedge 0 \leq \psi_\varepsilon(x) - \varphi_\varepsilon(x) < \varepsilon$$

**Theorem** (Simple approximation theorem). Let  $f : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$ ,  $E$  measurable. Then  $f$  is measurable iff there exists sequence  $(\varphi_n)$  of simple functions on  $E$  which converge pointwise on  $E$  to  $f$  and satisfy

$$\forall n \in \mathbb{N}, \forall x \in E, |\varphi_n|(x) \leq |f|(x)$$

If  $f$  is nonnegative,  $(\varphi_n)$  can be chosen to be increasing.

**Definition.** Let  $f, g : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$ . Then  $f = g$  **almost everywhere** if  $\{x \in E : f(x) \neq g(x)\}$  has measure 0.

**Proposition.** Let  $f_1, f_2, f_3 : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$  measurable. If  $f_1 = f_2$  almost everywhere and  $f_2 = f_3$  almost everywhere then  $f_1 = f_3$  almost everywhere.

**Remark.** Lebesgue measurable functions can be modified arbitrarily on a set of measure 0 without affecting measurability.

**Proposition.** Let  $f_n : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$  sequence of measurable functions,  $f : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$  measurable. Set of points where  $(f_n)$  converges pointwise to  $f$  is measurable.

**Proposition.** Let  $f, g : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$  measurable and finite almost everywhere on  $E$ .

- **Linearity:**  $\forall \alpha, \beta \in \mathbb{R}$ , there exists function equal to  $\alpha f + \beta g$  almost everywhere on  $E$  (any such function is measurable).
- **Products:** there exists function equal to  $fg$  almost everywhere on  $E$  (any such function is measurable).

**Definition.** Sequence of functions  $(f_n)$  with domain  $E$  **converge in measure** to  $f$  if  $(f_n)$  and  $f$  are finite almost everywhere and

$$\forall \varepsilon > 0, \quad \mu(\{x \in E : |f_n(x) - f(x)| > \varepsilon\}) \rightarrow 0 \text{ as } n \rightarrow \infty$$

## 5. The Lebesgue integral

### 5.1. The integral of a simple measurable function

**Definition.** Let  $\varphi$  be real-valued function taking finitely many values  $\alpha_1 < \dots < \alpha_n$ , then **standard representation** of  $\varphi$  is

$$\varphi = \sum_{i=1}^n \alpha_i \mathbb{1}_{A_i}, \quad A_i = \varphi^{-1}(\{\alpha_i\})$$

**Lemma.** Let  $\varphi = \sum_{i=1}^m \beta_i \mathbb{1}_{B_i}$ ,  $B_i$  disjoint measurable collection,  $\beta_i \in \mathbb{R}$ , then  $\varphi$  is simple measurable. If  $\varphi$  takes value 0 outside a set of finite measure then

$$\sum_{i=1}^n \alpha_i \mu(A_i) = \sum_{i=1}^m \beta_i \mu(B_i)$$

where  $A_i$  in standard representation.

**Definition.** Let  $\varphi$  be simple nonnegative measurable function or simple measurable function taking value 0 outside set of finite measure. **Integral** of  $\varphi$  with respect to  $\mu$  is

$$\int \varphi = \sum_{i=1}^n \alpha_i \mu(A_i)$$

where  $\varphi = \sum_{i=1}^n \alpha_i \mathbb{1}_{A_i}$  is standard representation. Here, use convention  $0 \cdot \infty = 0$ . For measurable  $E \subseteq \mathbb{R}$ , define

$$\int_E \varphi = \int \mathbb{1}_E \varphi$$

**Example.**

- Let  $\varphi_2 = \mathbb{1}_{[0,2]} + \mathbb{1}_{[1,3]} = \mathbb{1}_{[0,1) \cup (2,3]} + 2\mathbb{1}_{[1,2]}$  so  $\int \varphi_2 = 4$ .
- Let  $\varphi_3 = \mathbb{1}_{\mathbb{R}}$ , then  $\int \varphi_3 = 1 \cdot \infty = \infty$ .
- Let  $\varphi_4 = \mathbb{1}_{(0,\infty)} + (-1)\mathbb{1}_{(-\infty,0)}$ . This can't be integrated.

- Let  $\varphi_5 = \mathbb{1}_{(-1,0)} + (-1)\mathbb{1}_{(0,1)}$ , then  $\int \varphi_5 = 0$ .

**Lemma.** Let  $B_1, \dots, B_m$  be measurable sets,  $\beta_1, \dots, \beta_m \in \mathbb{R} - \{0\}$ . Then  $\varphi = \sum_{i=1}^m \beta_i \mathbb{1}_{B_i}$  is simple measurable function. Also,

$$\mu\left(\bigcup_{i=1}^m B_i\right) < \infty \implies \sum_{i=1}^m \alpha_i \mu(A_i) = \sum_{i=1}^m \beta_i \mu(B_i)$$

where  $A_i$  in standard representation.

**Proposition.** Let  $\varphi, \psi$  be simple measurable functions:

- If  $\varphi, \psi$  take value 0 outside a set of finite measure, then  $\forall \alpha, \beta \in \mathbb{R}$ ,

$$\int (\alpha\varphi + \beta\psi) = \alpha \int \varphi + \beta \int \psi$$

- If  $\varphi, \psi$  nonnegative, then  $\forall \alpha, \beta \geq 0$ ,

$$\int (\alpha\varphi + \beta\psi) = \alpha \int \varphi + \beta \int \psi$$

- **Monotonicity:**

$$0 \leq \varphi \leq \psi \implies 0 \leq \int \varphi \leq \int \psi$$

**Corollary.** Let  $\varphi$  nonnegative simple function, then

$$\int \varphi = \sup \left\{ \int \psi : 0 \leq \psi \leq \varphi, \psi \text{ simple measurable} \right\}$$

**Lemma.** Let  $\varphi$  simple measurable nonnegative function.  $\varphi$  takes value 0 outside a set of finite measure iff  $\int \varphi < \infty$ . Also,  $\int \varphi = \infty$  iff there exist  $\alpha > 0$ , measurable  $A$  with  $\mu(A) = \infty$  and  $\forall x \in A, \varphi(x) \geq \alpha$ .

**Lemma.** Let  $\{E_n\}$  be ascending collection of measurable sets,  $\cup_{n \in \mathbb{N}} E_n = \mathbb{R}$ . Let  $\varphi$  be simple nonnegative measurable function. Then

$$\int_{E_n} \varphi \rightarrow \int \varphi \quad \text{as } n \rightarrow \infty$$

## 5.2. The integral of a nonnegative function

**Notation.** Let  $\mathcal{M}^+$  denote collection of nonnegative measurable functions  $f : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ .

**Definition.** **Support** of measurable function  $f$  with domain  $E$  is  $\text{supp}(f) := \{x \in E : f(x) \neq 0\}$ .

**Definition.** Let  $f \in \mathcal{M}^+$ . **Integral of  $f$  with respect to  $\mu$**  is

$$\int f := \sup \left\{ \int \varphi : 0 \leq \varphi \leq f, \varphi \text{ simple measurable} \right\} \in \mathbb{R} \cup \{\infty\}$$

For measurable set  $E$ , define

$$\int_E f := \int \mathbb{1}_E f$$

**Proposition** (Monotonicity). Let  $f, g$  measurable, nonnegative. If  $g \leq f$  then  $\int g \leq \int f$ . Let  $E, F$  measurable. If  $E \subseteq F$  then  $\int_E f \leq \int_F f$ .

**Theorem** (Monotone convergence theorem). Let  $(f_n)$  be sequence in  $\mathcal{M}^+$ . If  $(f_n)$  is increasing on measurable set  $E$  and converges pointwise to  $f$  on  $E$  then

$$\int_E f_n \rightarrow \int_E f \quad \text{as } n \rightarrow \infty$$

**Corollary.** Restriction of integral to nonnegative functions is linear:  $\forall f, g \in \mathcal{M}^+$ ,  $\forall \alpha \geq 0$ ,

$$\begin{aligned} \int (f + g) &= \int f + \int g \\ \int \alpha f &= \alpha \int f \end{aligned}$$

**Lemma** (Fatou's Lemma). Let  $(f_n)$  be sequence in  $\mathcal{M}^+$ , then

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n$$

**Lemma.** Let  $(f_n) \subset \mathcal{M}^+$ , then

$$\int \sum_{n \in \mathbb{N}} f_n = \sum_{n \in \mathbb{N}} \int f_n$$

**Proposition** (Chebyshev's inequality). Let  $f$  be nonnegative measurable function on  $E$ . Then

$$\forall \lambda > 0, \quad \mu(\{x \in E : f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int_E f$$

**Proposition.** Let  $f$  be nonnegative measurable function on  $E$ . Then

$$\int_E f = 0 \iff f = 0 \text{ almost everywhere on } E$$

### 5.3. Integration of measurable functions

**Notation.** Let  $\mathcal{M}$  denote set of measurable functions.

**Definition.**  $f \in \mathcal{M}^+$  is **integrable** if  $\int f < \infty$ . By Chebyshev's inequality, if  $f$  is integrable, then  $f$  is finite almost everywhere.

**Definition.** Let  $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{\pm\infty\}$  measurable function.  $f$  is **integrable** if  $\int f^+$  and  $\int f^-$  are finite. In this case, for any measurable set  $E$ , define

$$\int_E f := \int_E f^+ - \int_E f^-$$

Note that if  $f$  integrable then  $f^+ - f^-$  is well-defined.

**Proposition.** If  $f = f_1 - f_2$ ,  $f_1, f_2 \in \mathcal{M}^+$ ,  $f_1, f_2$  integrable, then

$$\int f^+ - \int f^- = \int f_1 - \int f_2$$

**Definition.**  $f \in \mathcal{M}$  is **integrable over  $E$**  ( $E$  is measurable) if  $\int_E f^+$  and  $\int_E f^-$  are finite (i.e.  $f \cdot \mathbb{1}_E$  is integrable).

**Theorem.**  $f \in \mathcal{M}$  is integrable iff  $|f|$  is integrable. If  $f$  integrable, then

$$\left| \int f \right| \leq \int |f|$$

**Corollary.** Let  $f, g \in \mathcal{M}$ ,  $|f| \leq |g|$ . If  $g$  integrable then  $|f|$  is integrable, and  $\int |f| \leq \int |g|$ .

**Example.**  $\sin$  is not integrable over  $\mathbb{R}$ , but is integrable over  $[0, 2\pi]$ , since  $|f_{[0,2\pi]}| \leq \mathbb{1}_{[0,2\pi]}$ .

**Theorem** (Linearity of Integration). Let  $f, g \in \mathcal{M}$  integrable. Then  $f + g$  is integrable and  $\forall \alpha \in \mathbb{R}$ ,  $\alpha f$  is integrable. The integral is linear:

$$\begin{aligned} \int (f + g) &= \int f + \int g \\ \int \alpha f &= \alpha \int f \end{aligned}$$

**Theorem** (Dominated Convergence Theorem). Let  $(f_n)$  be sequence of integrable functions. If there exists an integrable  $g$  with  $\forall n \in \mathbb{N}$ ,  $|f_n| \leq g$ , and  $f_n \rightarrow f$  pointwise almost everywhere then  $f$  is integrable and

$$\int f = \lim_{n \rightarrow \infty} \int f_n$$

## 5.4. Integrability: Riemann vs Lebesgue

**Proposition.** Let  $f$  bounded function on bounded measurable domain  $E$ . Then  $f$  is measurable and  $\int_E |f| < \infty$  iff

$$\sup \left\{ \int_E \varphi : \varphi \leq f, \varphi \text{ simple measurable} \right\} = \inf \left\{ \int_E \psi : f \leq \psi : \psi \text{ simple measurable} \right\}$$

(If  $f$  satisfies either condition then  $\int_E f$  is equal to the two above expressions).

**Definition.** Bounded function  $f$  is **Lebesgue integrable** if it satisfies either of the equivalences in the above proposition.

**Definition.** Let  $P = \{x_0, \dots, x_n\}$  partition of  $[a, b]$ ,  $f : [a, b] \rightarrow \mathbb{R}$  bounded. **Lower and upper Darboux sums** for  $f$  with respect to  $P$  are



$$L(f, P) := \sum_{i=1}^n m_i(x_i - x_{i-1}), \quad U(f, P) := \sum_{i=1}^n M_i(x_i - x_{i-1})$$

where

$$m_i := \inf\{f(x) : x \in (x_{i-1}, x_i)\}, \quad M_i := \sup\{f(x) : x \in (x_{i-1}, x_i)\}$$

If  $P \subseteq Q$  ( $Q$  is a **refinement of  $P$** ), then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$$

**Definition.** Lower and upper Riemann integrals of  $f$  over  $[a, b]$  are

$$\underline{\mathcal{J}}_a^b(f) := \sup\{L(f, P) : P \text{ partition of } [a, b]\}$$

$$\overline{\mathcal{J}}_a^b(f) := \inf\{U(f, P) : P \text{ partition of } [a, b]\}$$

**Definition.** Let  $f : [a, b] \rightarrow \mathbb{R}$  bounded, then  $f$  is **Riemann integrable** ( $f \in \mathcal{R}$ ), if

$$\underline{\mathcal{J}}_a^b(f) = \overline{\mathcal{J}}_a^b(f)$$

and common value  $\mathcal{J}_a^b(f) = \int_a^b f(x) dx$  is **Riemann integral** of  $f$ .

**Remark.** Let  $g : [a, b] \rightarrow \mathbb{R}$  step function with discontinuities at  $P = \{x_0, \dots, x_n\}$ , so  $g = \sum_{i=1}^n \alpha_i \mathbb{1}_{(x_{i-1}, x_i)}$  almost everywhere. So  $g$  is simple measurable and

$$L(g, P) = \sum_{i=1}^n \alpha_i(x_i - x_{i-1}) = U(g, P) = \int g = \mathcal{J}_a^b(g)$$

Hence for any bounded  $f : [a, b] \rightarrow \mathbb{R}$ ,

$$\underline{\mathcal{J}}_a^b(f) = \sup\left\{\int \varphi : \varphi \leq f, \varphi \text{ step function}\right\},$$

$$\overline{\mathcal{J}}_a^b(f) = \inf\left\{\int \psi : f \leq \psi, \psi \text{ step function}\right\}$$

**Theorem.** Let  $f : [a, b] \rightarrow \mathbb{R}$  bounded,  $a, b \neq \pm\infty$ . If  $f$  Riemann integrable over  $[a, b]$  then  $f$  Lebesgue integrable over  $[a, b]$  and the two integrals are equal.

**Theorem.** Let  $f : [a, b] \rightarrow \mathbb{R}$  bounded,  $a, b \neq \pm\infty$ . Then  $f$  is Riemann integrable on  $[a, b]$  iff  $f$  is continuous on  $[a, b]$  except on a set of measure zero.

**Lemma.** Let  $(\varphi_n), (\psi_n)$  be sequences of functions, all integrable over  $E$ ,  $(\varphi_n)$  increasing on  $E$ ,  $(\psi_n)$  decreasing on  $E$ . Let  $f : E \rightarrow \mathbb{R}$  with

$$\forall n \in \mathbb{N}, \varphi_n \leq f \leq \psi_n \text{ on } E, \quad \lim_{n \rightarrow \infty} \int_E (\psi_n - \varphi_n) = 0$$

Then  $\varphi_n, \psi_n \rightarrow f$  pointwise almost everywhere on  $E$ ,  $f$  is integrable over  $E$  and

$$\lim_{n \rightarrow \infty} \int_E \varphi_n = \lim_{n \rightarrow \infty} \int_E \psi_n = \int_E f$$

**Definition.** For partition  $P = \{x_0, \dots, x_n\}$ , **gap** of  $P$  is

$$\text{gap}(P) := \max\{|x_i - x_{i-1}| : i \in \{1, \dots, n\}\}$$

**Lemma.** Let  $f : [a, b] \rightarrow \mathbb{R}$ ,  $E \subseteq [a, b]$  be set where  $f$  is continuous. Let  $(P_n)$  be sequence of partitions of  $[a, b]$  with  $P_{n+1} \subseteq P_n$  and  $\text{gap}(P_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Let  $\varphi_n, \psi_n : [a, b] \rightarrow \mathbb{R}$  step functions with

$$\varphi_n(x) := \inf\{f(x) : x \in (x_{i-1}, x_i)\}, \quad \psi_n(x) := \sup\{f(x) : x \in (x_{i-1}, x_i)\}$$

for  $P_n = \{x_0, \dots, x_n\}$ . Then  $\forall x \in E - \cup_{n \in \mathbb{N}} P_n$ ,

$$\varphi_n(x), \psi_n(x) \rightarrow f(x) \quad \text{as } n \rightarrow \infty$$

**Definition.** Let  $f : (a, b) \rightarrow \mathbb{R}$ ,  $-\infty \leq a < b < \infty$ ,  $f$  bounded and Riemann integrable on all closed bounded sub-intervals of  $(a, b)$ . If

$$\lim_{t \rightarrow a, t > a} \mathcal{J}_t^b(f)$$

exists then this is defined as the **improper Riemann integral**  $\mathcal{J}_a^b(f)$ . Similar definitions exist for  $f : (a, b) \rightarrow \mathbb{R}$  and  $f : [a, b) \rightarrow \mathbb{R}$ .

**Note.** Improper Riemann integral may exist without function being Lebesgue integral.

**Proposition.** If  $f$  is integrable, the improper Riemann integral is equal to the Lebesgue integral whenever the former exists.

**Definition.** Let  $\alpha : [a, b] \rightarrow \mathbb{R}$  monotonically increasing (and so bounded). For partition  $P = \{x_0, \dots, x_n\}$  of  $[a, b]$  and bounded  $f : [a, b] \rightarrow \mathbb{R}$ , define

$$L(f, P, \alpha) := \sum_{i=1}^n m_i(\alpha(x_i) - \alpha(x_{i-1})), \quad U(f, P, \alpha) := \sum_{i=1}^n M_i(\alpha(x_i) - \alpha(x_{i-1}))$$

where  $m_i := \inf\{f(x) : x \in (x_{i-1}, x_i)\}$ ,  $M_i := \sup\{f(x) : x \in (x_{i-1}, x_i)\}$ . Then  $f$  is **integrable with respect to  $\alpha$** ,  $f \in \mathcal{R}(\alpha)$ , if

$$\inf\{U(f, P, \alpha) : P \text{ partition of } [a, b]\} = \sup\{L(f, P, \alpha) : P \text{ partition of } [a, b]\}$$

and the common value  $\int_a^b f d\alpha$  is the **Riemann-Stieltjes integral** of  $f$  with respect to  $\alpha$ .

**Proposition.** Let  $f : (a, b) \rightarrow \mathbb{R}$ , then set of points where  $f$  is differentiable is measurable.

**Remark.** If  $\alpha : [0, 1] \rightarrow [a, b]$  bijection, then

$$\int_0^1 f \circ \alpha d\alpha = \int_a^b f(x) dx$$

**Proposition.** Let  $\alpha$  be monotonically increasing and differentiable with  $\alpha' \in \mathcal{R}$ . Then  $g \in \mathcal{R}(\alpha)$  iff  $g\alpha' \in \mathcal{R}$ , and in that case,

$$\int_a^b g \, d\alpha = \int_a^b g(x)\alpha'(x) \, dx$$

**Remark.** When  $g = 1$ , this says  $\int_a^b 1 \, d\alpha = \alpha(b) - \alpha(a) = \int \alpha'(x) \, dx$ , similar to the fundamental theorem of calculus.

## 6. Lebesgue spaces

### 6.1. Normed linear spaces

**Definition.** Let  $X$  be **complex linear space** (vector space over  $\mathbb{C}$ ).  $\|\cdot\| : X \rightarrow \mathbb{R}_{\geq 0}$  is **norm on  $X$**  if

- $\forall x \in X, \|x\| = 0 \iff x = 0$ .
- $\forall x \in X, \forall \lambda \in \mathbb{C}, \|\lambda x\| = |\lambda| \|x\|$ .
- $\forall x, y \in X, \|x + y\| \leq \|x\| + \|y\|$ .

$X$  equipped with norm  $\|\cdot\|$ ,  $(X, \|\cdot\|)$ , is called **complex normed linear space**.

**Example.**

- $\|x\| = \sqrt{x\bar{x}}$  is norm on  $\mathbb{C}$ .
- Let  $C[a, b]$  denote linear space of continuous real-valued functions on  $[a, b]$ . Then

$$\|f\|_{\max} := \max\{|f(x)| : x \in [a, b]\}$$

is norm on  $C[a, b]$ .

**Proposition.** Norm induces metric on  $X$ :  $d(x, y) = \|x - y\|$ .

**Definition.** Let  $(X, \|\cdot\|)$  be normed linear space.

- Sequence  $(f_n)$  in  $X$  is **Cauchy sequence** in  $X$  if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n, m \geq N, \|f_n - f_m\| < \varepsilon$$

- Sequence  $(f_n)$  in  $X$  **converges in  $X$** ,  $\|f_n - f\| \rightarrow 0$  as  $n \rightarrow \infty$ , if

$$\exists f \in X : \forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \|f_n - f\| < \varepsilon$$

- $(X, \|\cdot\|)$  is **complete** if every Cauchy sequence converges in  $X$ .
- **Banach space** is complete normed linear space.

**Proposition.** Let  $(X, \|\cdot\|)$  be normed linear space.

- If  $(x_n)$  converges in  $X$ ,  $(x_n)$  is Cauchy sequence in  $X$ .
- Let  $(x_n)$  be Cauchy sequence in  $X$ . If  $(x_n)$  has convergent subsequence in  $X$  then  $(x_n)$  converges in  $X$ .

### 6.2. Lebesgue spaces $L^p$ , $p \in [1, \infty)$

**Definition.** Let  $p \in [1, \infty)$ ,  $E \subseteq \mathbb{R}$ .

- Linear space  $L^p(E)$  is defined as

$$L^p(E) := \left\{ f : E \rightarrow \mathbb{C} : f \text{ is measurable and } \int_E |f|^p < \infty \right\} / \cong$$

where  $f \cong g$  iff  $f = g$  almost everywhere:

$$f \cong g \iff \exists F \subseteq E : \mu(F) = 0 \wedge \forall x \in E - F, f(x) = g(x)$$

- Define  $\|\cdot\|_{L^p} : L^p(E) \rightarrow \mathbb{R}$  as

$$\|f\|_{L^p} := \left( \int_E |f|^p \right)^{1/p}$$

**Remark.**

- We often consider space  $L^p(E)$  of real-valued measurable functions  $f : E \rightarrow \mathbb{R}$  such that  $\int_E |f|^p < \infty$ .
- For  $f : E \rightarrow \mathbb{C}$ ,  $f = f_1 + if_2$ ,  $f$  is measurable iff  $f_1 : E \rightarrow \mathbb{R}$  and  $f_2 : E \rightarrow \mathbb{R}$  are measurable. Also,

$$\int_E |f|^p < \infty \iff \left( \int_E |f_1|^p < \infty \wedge \int_E |f_2|^p < \infty \right)$$

**Example.** Let  $E = \mathbb{R}$ ,  $f(x) = \mathbb{1}_{\mathbb{R}-\mathbb{Q}}(x) + i\mathbb{1}_{\mathbb{Q}}(x)$  and  $g(x) = 1$ . Then  $\mu(\mathbb{Q}) = 0$  so  $f \cong g$ .

**Proposition.** Let  $(f_n), (g_n)$  sequences of measurable functions,  $\forall n \in \mathbb{N}, f_n \cong g_n$ ,  $\lim_{n \rightarrow \infty} f_n = f$  and  $\lim_{n \rightarrow \infty} g_n = g$ . Then  $f \cong g$ .

**Definition.**  $p, q \in \mathbb{R}$  are **conjugate exponents** if  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Lemma** (Young's inequality). Let  $p, q$  conjugate exponents, then

$$\forall A, B \in \mathbb{R}_{\geq 0}, \quad AB \leq \frac{A^p}{p} + \frac{B^q}{q}$$

with equality iff  $A^p = B^q$ .

**Lemma** (Hölder's inequality). Let  $p, q$  conjugate exponents. If  $f \in L^p(E)$ ,  $g \in L^q(E)$ , then

$$\int_E |fg| \leq \|f\|_{L^p} \|g\|_{L^q}$$

**Corollary** (Cauchy-Schwarz inequality for  $L^2(E)$ ). If  $f, g \in L^2(E)$ , then

$$\left| \int_E f\bar{g} \right| \leq \int_E |fg| \leq \|f\|_{L^2} \|g\|_{L^2}$$

**Lemma** (Minkowski's inequality). Let  $p \in [1, \infty)$ . If  $f, g \in L^p(E)$  then  $f + g \in L^p(E)$  and

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

**Theorem.** For  $p \in [1, \infty)$ ,  $(L^p(E), \|\cdot\|_{L^p})$  is normed linear space.

**Proposition.** Let  $1 \leq p < q < \infty$ . If  $\mu(E) < \infty$  then  $L^q(E) \subseteq L^p(E)$  and

$$\|f\|_{L^p} \leq \mu(E)^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^q}$$

**Remark.**

- Convergence in  $L^p$  is also called convergence in the mean of order  $p$ .
- This notion of convergence is different to pointwise convergence, uniform convergence and convergence in measure.

**Theorem** (Riesz-Fischer). For  $p \in [1, \infty)$ ,  $(L^p(E), \|\cdot\|_{L^p})$  is complete.

### 6.3. Lebesgue space $L^\infty$

**Definition.**

- Let  $f : E \rightarrow \mathbb{C}$  measurable.  $f$  is **essentially bounded** if

$$\exists M \geq 0 : |f(x)| \leq M \quad \text{almost everywhere on } E$$

- $L^\infty(E)$  is collection of equivalence classes of essentially bounded functions where  $f \cong g$  iff  $f = g$  almost everywhere.
- For  $f \in L^\infty(E)$ , define

$$\|f\|_{L^\infty} := \text{ess sup}|f| := \inf\{M \in \mathbb{R} : \mu(\{x \in E : |f(x)| > M\}) = 0\}$$

**Proposition.**

- $0 \leq |f(x)| \leq \|f\|_{L^\infty}$  almost everywhere.
- $\|f\|_{L^\infty}$  is norm on  $L^\infty(E)$ .
- If  $f \in L^1(E)$ ,  $g \in L^\infty(E)$ , then

$$\int_E |fg| \leq \|f\|_{L^1} \|g\|_{L^\infty}$$

**Proposition.** Let  $(f_n)$  sequence of functions in  $L^\infty(E)$ . Then  $(f_n)$  converges to  $f \in L^\infty(E)$  iff there exists  $G \subseteq E$  with  $\mu(G) = 0$  and  $(f_n)$  converges to  $f$  uniformly on  $E - G$ .

**Theorem.**  $(L^\infty(E), \|\cdot\|_{L^\infty})$  is complete.

**Remark.** If  $\mu(E) < \infty$ , then  $L^\infty(E) \subset L^p(E)$  for  $p \in [1, \infty)$  and

$$\|f\|_{L^p} \leq \mu(E)^{1/p} \|f\|_{L^\infty}$$

since

$$\|f\|_{L^p}^p = \int_E |f|^p \leq \int_E \|f\|_{L^\infty}^p \cdot \mathbf{1}_E = \|f\|_{L^\infty}^p \mu(E)$$

### 6.4. Approximation and separability

**Definition.** Let  $(X, \|\cdot\|)$  be normed linear space. Let  $F \subseteq G \subseteq X$ .  $F$  is **dense in  $G$**  if

$$\forall g \in G, \forall \varepsilon > 0, \exists f \in F : \|f - g\| < \varepsilon$$

**Proposition.**

- $F$  is dense in  $G$  iff for every  $g \in G$ , there exists sequence  $(f_n)$  in  $F$  such that  $\lim_{n \rightarrow \infty} f_n = g$  in  $X$ .
- For  $F \subseteq G \subseteq H \subseteq X$ , if  $F$  dense in  $G$  and  $G$  dense in  $H$ , then  $F$  dense in  $H$ .

**Proposition.** Let  $p \in [1, \infty]$ . Then subspace of simple functions in  $(L^p(E), \|\cdot\|_{L^p})$  is dense in  $(L^p(E), \|\cdot\|_{L^p})$ .

**Definition.**  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  is **step function** if it can be written as

$$\psi = \sum_{k=1}^N \tilde{a}_k \mathbb{1}_{(a_k, b_k)}$$

where the intervals  $(a_k, b_k)$  are disjoint.

**Proposition.** Let  $[a, b]$  be bounded,  $p \in [1, \infty)$ . Then subspace of step functions on  $[a, b]$  is dense in  $(L^p([a, b]), \|\cdot\|_{L^p})$ .

**Definition.** Normed linear space  $(X, \|\cdot\|)$  is **separable** if there exists countable, dense subset  $X' \subseteq X$ .

**Example.**  $\mathbb{R}$  is separable, since  $\mathbb{Q}$  is countable and dense in  $\mathbb{R}$ .

**Theorem.** Let  $E \subseteq \mathbb{R}$  measurable,  $p \in [1, \infty)$ . Then  $(L^p(E), \|\cdot\|_{L^p})$  is separable. In particular, step functions are dense in  $L^p(E)$  for  $p \in [1, \infty)$ .

**Proposition.** Let  $\varepsilon > 0$ ,  $f \in L^p(E)$ ,  $p \in [1, \infty)$ . There exists continuous  $g \in L^p(E)$  such that  $\|f - g\|_{L^p} < \varepsilon$ .

**Remark.** Linear space of continuous functions that vanish outside bounded set is dense in  $(L^p(E), \|\cdot\|_{L^p})$  for  $p \in [1, \infty)$ .

**Remark.** Differentiable functions are also dense in  $(L^p(E), \|\cdot\|_{L^p})$  for  $p \in [1, \infty)$ .

**Remark.** Step functions and continuous functions are not dense in  $(L^\infty(E), \|\cdot\|_{L^\infty})$ .

**Example.** In general,  $(L^\infty(E), \|\cdot\|_{L^\infty})$  is not separable. Let  $[a, b]$  be bounded,  $a \neq b$ . Assume there is countable  $\{f_n : n \in \mathbb{N}\}$  which is dense in  $(L^\infty([a, b]), \|\cdot\|_{L^\infty})$ . Then for every  $x \in [a, b]$ , can choose  $g(x) \in \mathbb{N}$  such that

$$\|\mathbb{1}_{[a, x]} - f_{g(x)}\|_{L^\infty} < \frac{1}{2}$$

Also, for  $x_1 \leq x_2$ ,

$$\|\mathbb{1}_{[a, x_1]} - \mathbb{1}_{[a, x_2]}\|_{L^\infty} = \begin{cases} 1 & \text{if } a \leq x_1 < x_2 \leq b \\ 0 & \text{if } x_1 = x_2 \end{cases}$$

and

$$\begin{aligned} \|\mathbb{1}_{[a, x_1]} - \mathbb{1}_{[a, x_2]}\|_{L^\infty} &\leq \|\mathbb{1}_{[a, x_1]} - f_{g(x_1)}\|_{L^\infty} + \|f_{g(x_1)} - f_{g(x_2)}\|_{L^\infty} + \|f_{g(x_2)} - \mathbb{1}_{[a, x_2]}\|_{L^\infty} \\ &< 1 + \|f_{g(x_1)} - f_{g(x_2)}\|_{L^\infty} \end{aligned}$$

If  $g(x_1) = g(x_2)$  then  $\|\mathbb{1}_{[a, x_1]} - \mathbb{1}_{[a, x_2]}\|_{L^\infty} = 0$  so  $g : [a, b] \rightarrow \mathbb{N}$  is injective. But  $\mathbb{N}$  is countable and  $[a, b]$  is not countable: contradiction.

## 6.5. Riesz representation theorem for $L^p(E)$ , $p \in [1, \infty)$

**Definition.** Let  $X$  be linear space.  $T : X \rightarrow \mathbb{R}$  is **linear functional** if

$$\forall f, g \in X, \forall a, b \in \mathbb{R}, \quad T(af + bg) = aT(f) + bT(g)$$

Any linear combination of linear functionals is linear, so set of linear functionals on linear space is also linear space.

**Definition.** Let  $(X, \|\cdot\|)$  be normed linear space.  $T : X \rightarrow \mathbb{R}$  is **bounded functional** if

$$\exists M \geq 0 : \forall f \in X, \quad |T(f)| \leq M\|f\|$$

**Norm** of  $T$ ,  $\|T\|_*$ , is the smallest such  $M$ .

**Remark.** For bounded linear functional  $T$  on normed linear space  $(X, \|\cdot\|)$ ,

$$|T(f) - T(g)| \leq \|T\|_* \|f - g\|$$

This gives the following continuity property: if  $f_n \rightarrow f \in X$ , then  $T(f_n) \rightarrow T(f)$ .

**Example.** Let  $E \subseteq \mathbb{R}$  measurable,  $p \in [1, \infty)$ ,  $q$  conjugate to  $p$ . Let  $h \in L^q(E)$ . Define  $T : L^p(E) \rightarrow \mathbb{R}$  by

$$T(f) = \int_E h \cdot f$$

By Holder's inequality,

$$|T(f)| = \left| \int_E hf \right| \leq \int_E |hf| \leq \|h\|_{L^q} \|f\|_{L^p}$$

So  $T$  is bounded linear functional.

**Remark.** We can write  $\|\cdot\|_*$  as

$$\|T\|_* := \inf\{M \in \mathbb{R} : \forall f \in X, |T(f)| \leq M\|f\|\} = \sup\{|T(f)| : f \in X, \|f\| \leq 1\}$$

**Definition.** **Dual space** of  $X$ ,  $X^*$ , is set of bounded linear functionals on  $X$  with norm  $\|\cdot\|_*$ .

**Proposition.** Let  $(X, \|\cdot\|)$  be normed linear space, then dual space of  $X$  is linear space with norm  $\|\cdot\|_*$ .

**Remark.** Bounded linear functional is special case of **bounded linear transformation** between normed spaces.  $T : X \rightarrow Y$  is bounded linear transformation if  $T(af + bg) = aT(f) + bT(g)$  and  $\exists M \geq 0 : \|T(f)\|_Y \leq M\|f\|_X$ .

**Proposition.** Let  $E \subseteq \mathbb{R}$  measurable,  $p \in [1, \infty)$ ,  $q$  conjugate to  $p$ ,  $h \in L^q(E)$ . Define  $T : L^p(E) \rightarrow \mathbb{R}$  by

$$T(f) = \int_E hf$$

Then  $\|T\|_* = \|h\|_{L^q}$ .

**Theorem** (Riesz representation theorem for  $L^p$ ). Let  $p \in [1, \infty)$ ,  $q$  conjugate to  $p$ ,  $E \subseteq \mathbb{R}$  measurable. For  $h \in L^q(E)$ , define bounded linear functional  $R_h : L^p(E) \rightarrow \mathbb{R}$  by

$$R_h(f) = \int_E hf$$

Then for every bounded linear functional  $T : L^p(E) \rightarrow \mathbb{R}$ , there is unique  $h \in L^q(E)$  such that

$$R_h = T \quad \wedge \quad \|T\|_* = \|h\|_{L^q}$$

**Theorem.** Let  $[a, b]$  be non-degenerate, bounded interval,  $p \in [1, \infty)$ ,  $q$  conjugate to  $p$ . If  $T$  is bounded linear functional on  $L^p([a, b])$  then there exists  $h \in L^q([a, b])$  such that

$$T(f) = \int_a^b hf$$

## 7. Hilbert spaces

### 7.1. Inner product spaces

**Definition.** Let  $H$  be complex linear space. **Inner product** on  $H$  is function  $\langle \cdot, \cdot \rangle : H \times H \rightarrow \mathbb{C}$  such that  $\forall a, b \in \mathbb{C}, \forall x, y, z \in H$ ,

- **Linear in first variable:**  $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ .
- **Conjugate symmetric:**  $\langle x, y \rangle = \overline{\langle y, x \rangle}$ .
- **Positive:**  $x \neq 0 \implies \langle x, x \rangle \in (0, \infty)$
- $\langle x, x \rangle = 0 \iff x = 0$ .

These imply that  $\langle 0, x \rangle = 0$  and inner product is conjugate linear in second variable:  $\langle z, ax + by \rangle = \bar{a}\langle z, x \rangle + \bar{b}\langle z, y \rangle$ .

**Example.**

- $\mathbb{R}^n$  has inner product  $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ .
- $\mathbb{C}^n$  has inner product  $\langle x, y \rangle = \sum_{i=1}^n x_i \bar{y}_i$ .
- Inner product induces metric on  $H$ :

$$d(x, y) = \langle x - y, x - y \rangle^{1/2}$$

**Definition.** Complex linear space  $H$  with inner product  $\langle \cdot, \cdot \rangle$  is called **pre-Hilbert space** or **inner product space**.

**Definition.** Let  $H$  inner product space. For  $x \in H$ , define the norm

$$\|x\| = \sqrt{\langle x, x \rangle}$$

**Proposition.**  $\|x \pm y\|^2 = \|x\|^2 \pm 2 \operatorname{Re}(\langle x, y \rangle) + \|y\|^2$ .

**Theorem** (Cauchy-Schwarz inequality). Let  $(H, \langle \cdot, \cdot \rangle)$  be pre-Hilbert space. Then

$$\forall x, y \in H, \quad |\langle x, y \rangle| \leq \|x\| \|y\|$$



with equality iff  $x$  and  $y$  linearly dependent.

**Theorem** (Parallelogram Identity). A normed linear space  $X$  is an inner product space with norm derived from the inner product (i.e.  $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ ) iff

$$\forall x, y \in X, \quad \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

**Definition.** Let  $(X, \langle \cdot, \cdot \rangle_X)$ ,  $(Y, \langle \cdot, \cdot \rangle_Y)$  be inner product spaces.

- An inner product on  $X \times Y$  is

$$\langle (x_1, y_1), (x_2, y_2) \rangle_{X \times Y} = \langle x_1, x_2 \rangle_X + \langle y_1, y_2 \rangle_Y$$

- The associated norm on  $X \times Y$  is

$$\|(x, y)\|_{X \times Y} = \sqrt{\langle (x, y), (x, y) \rangle_{X \times Y}} = \sqrt{\|x\|_X^2 + \|y\|_Y^2}$$

**Theorem.** Let  $X$  inner product space,  $x_n \rightarrow x$ ,  $y_n \rightarrow y$  in  $X$ . Then  $\langle x_n, y_n \rangle_X \rightarrow \langle x, y \rangle_X$ .

*Proof.* Use  $|\langle x_n, y_n \rangle - \langle x, y \rangle| = |\langle x_n - x, y_n \rangle + \langle x, y_n \rangle - \langle x, y \rangle + \langle x, y_n - y \rangle|$  and Cauchy-Schwarz, reverse triangle inequality to show  $\|y_n\| \rightarrow \|y\|$ . □

**Proposition.** The norm and inner product are continuous.

## 7.2. Hilbert spaces

**Definition.** Hilbert space is inner product space which is complete with respect to norm induced by inner product.

**Example.**  $\mathbb{R}^n$  with standard inner product is Hilbert space.

**Example.** Define inner product on  $L^2(E)$

$$\langle f, g \rangle_{L^2} := \int_E f \bar{g}$$

Induced norm is the  $L^2$  norm. So by Riesz-Fischer theorem,  $(L^2(E), \langle \cdot, \cdot \rangle_{L^2})$  is Hilbert space.

**Definition.** Let  $H$  Hilbert space with inner product  $\langle \cdot, \cdot \rangle$ .

- $x, y \in H$  are **orthogonal**,  $x \perp y$  if  $\langle x, y \rangle = 0$ .
- $A, B \subseteq H$  are **orthogonal**,  $A \perp B$  if  $\forall x \in A, \forall y \in B, \quad x \perp y$ .
- **Orthogonal complement** of  $A \subseteq H$  is

$$A^\perp := \{x \in H : \forall y \in A, \quad x \perp y\}$$

**Theorem** (Pythagorean Theorem). If  $x_1, \dots, x_n \in H$ ,  $x_i \perp x_j$  for  $i \neq j$ , then

$$\left\| \sum_{i=1}^n x_i \right\|^2 = \sum_{i=1}^n \|x_i\|^2$$

*Proof.* Use linearity of inner product and orthogonal condition. □

**Theorem.** Let  $H$  Hilbert space,  $A \subseteq H$ , then  $A^\perp$  is closed subspace of  $H$ .

*Proof.*

- Subspace:
  - For  $y, z \in A^\perp$ ,  $\lambda, \mu \in \mathbb{C}$ , show  $\forall x \in A$ ,  $\lambda y + \mu z \in A^\perp$ .
- Closed:
  - Show if  $(y_n) \subseteq A^\perp$ ,  $y_n \rightarrow y$ , then  $y \in A^\perp$ :
    - Let  $x \in A$ , then show  $|\langle x, y \rangle| \rightarrow 0$  by squeezing, triangle inequality and Cauchy-Schwarz.

□

**Theorem (Projection).** Let  $M$  closed subspace of Hilbert space  $H$ .

- For every  $x \in H$ , there exists unique closest point  $y \in M$ :

$$\forall x \in H, \exists! y \in M : \|x - y\| = \min\{\|x - z\| : z \in M\}$$

We say  $y$  is “the best approximation” to  $x$  in  $M$ .

- The point  $y \in M$  closest to  $x \in H$  is unique element of  $M$  such that  $(x - y) \perp M$ .

*Proof.*

- Let  $d = \inf\{\|x - z\| : z \in M\}$ . Show that  $\exists y \in M : \|x - y\| = d$ :
  - There is sequence  $(y_n) \subset M$  with  $\|x - y_n\| \rightarrow d$ . Show that  $(y_n)$  is Cauchy:
    - $\|y_m - y_n\|^2 + \|2x - y_m - y_n\|^2 = 2\|x - y_m\|^2 + 2\|x - y_n\|^2$  by parallelogram identity.
    - $\frac{y_m + y_n}{2} \in M$ , so  $\|2x - y_m - y_n\| \geq 2d$ .
  - Deduce that  $y_n \rightarrow y \in M$  and  $\|x - y\| \rightarrow d$  by squeezing.
- Uniqueness of  $y$ :
  - Let  $\|x - y\| = d = \|x - y'\|$ .
  - By parallelogram identity,  $2\|x - y\|^2 + 2\|x - y'\|^2 = \|2x - y - y'\|^2 + \|y - y'\|^2$ .
  - Use that  $\frac{y + y'}{2} \in M$  to show  $\|y - y'\| = 0$ .
- To show  $z = x - y \perp M$ :
  - For  $w \in M$ , write  $\langle z, w \rangle = |\langle z, w \rangle| \lambda$  where  $\lambda = e^{i\theta}$ , set  $u = \lambda w$ .
  - Define  $f(t) = \|z + tu\|^2$ , show  $t = 0$  is minimum of  $f$  and so  $0 = f'(0)$ , hence  $z \in M^\perp$ .
- To show uniqueness of  $z$ :
  - Show for  $y, y' \in M$  such that  $x - y \perp M$  and  $x - y' \perp M$ , then  $\langle y - y', w \rangle = 0$  for any  $w \in M$ . Set  $w = y - y'$  to give  $y = y'$ .

□

**Definition.** Direct sum of subspaces  $M$  and  $N$  of linear space is

$$M \oplus N := \{y + z : y \in M, z \in N\}$$

**Corollary.** If  $M$  closed subspace of Hilbert space  $H$ , then  $H = M \oplus M^\perp$ .

For all  $x \in H$ ,  $x$  can be written uniquely as  $x = y + z$  where  $y$  is best approximation to  $x$  in  $M$  and  $z = x - y \perp M$ .

*Proof.* By above theorem.

□

**Definition.** Let  $H$  Hilbert space.  $\{u_\alpha\}_{\alpha \in I}$  is **orthonormal** if it is **orthogonal**:  $u_\alpha \perp u_\beta$  for  $\alpha \neq \beta$ , and **normalised**:  $\forall \alpha \in I, \|u_\alpha\| = 1$ .

**Definition.** Let  $X$  Banach space,  $\{x_\alpha \in X : \alpha \in I\}$  be indexed set where  $I$  is countable or uncountable.

- For each finite  $J \subseteq I$ , define **partial sum** as

$$S_J := \sum_{\alpha \in J} x_\alpha$$

- Unordered sum of  $\{x_\alpha \in X : \alpha \in I\}$  **converges unconditionally** to  $x \in X$ , written  $x = \sum_{\alpha \in I} x_\alpha$ , if  $\forall \varepsilon > 0$ , there exists finite  $J \subseteq I$  such that  $\|S_K - x\| < \varepsilon$  for every finite  $J \subseteq K \subseteq I$ .
- Unordered sum  $\sum_{\alpha \in I} x_\alpha$  is **Cauchy** if  $\forall \varepsilon > 0$ , there exists finite  $J \subseteq I$  such that  $\|S_L\| < \varepsilon$  for every finite  $L \subseteq I - J$ . Note that

$$\|S_L\| = \left\| \sum_{\alpha \in L \cup J} x_\alpha - \sum_{\alpha \in J} x_\alpha \right\|$$

- Unordered sum of  $\{x_\alpha \in X : \alpha \in I\}$  **converges absolutely** if  $\sum_{\alpha \in I} \|x_\alpha\|$  converges unconditionally in  $\mathbb{R}$ .

**Proposition.** Unordered sum in Banach space converges unconditionally iff it is Cauchy.

**Definition.** Let  $\{c_\alpha : \alpha \in I\} \subseteq [0, \infty]$ . Define

$$\sum_{\alpha \in I} c_\alpha = \sup \left\{ \sum_{\alpha \in J} c_\alpha : J \subseteq I, J \text{ finite} \right\}$$

**Proposition.** Let  $\{c_\alpha : \alpha \in I\} \subseteq [0, \infty]$ ,  $K = \{\alpha \in I : c_\alpha > 0\}$ . If  $\sum_{\alpha \in I} c_\alpha < \infty$ , then  $K$  is countable.

**Theorem** (Bessel's inequality). Let  $U = \{u_\alpha : \alpha \in I\}$  orthonormal in Hilbert space  $H$ . Then

$$\forall x \in H, \quad \sum_{\alpha \in I} |\langle x, u_\alpha \rangle|^2 \leq \|x\|^2$$

In particular,  $\forall x \in H, \{\alpha \in I : \langle x, u_\alpha \rangle \neq 0\}$  is countable.

*Proof.*

- Prove for any finite  $J \subseteq I$ , then take supremum on LHS.
- Show that

$$\left\| x - \sum_{\alpha \in J} \langle x, u_\alpha \rangle u_\alpha \right\|^2 = \|x\|^2 - \sum_{\alpha \in J} |\langle x, u_\alpha \rangle|^2$$

using equation 2.2 and Pythagorean theorem.

□

**Theorem.** If  $U = \{u_\alpha : \alpha \in I\}$  is orthonormal subset of Hilbert space  $H$  then the following are equivalent:

- If  $\forall \alpha \in I, \langle x, u_\alpha \rangle = 0$ , then  $x = 0$ .
- $\forall x \in H, x = \sum_{\alpha \in I} \langle x, u_\alpha \rangle u_\alpha$  where sum converges unconditionally in  $H$  and only has countably many non-zero terms.
- **Parseval's identity:**

$$\forall x \in H, \quad \|x\|^2 = \sum_{\alpha \in I} |\langle x, u_\alpha \rangle|^2$$

*Proof.*

- (i)  $\implies$  (ii): let  $\{\alpha_j : j \in \mathbb{N}\}$  be set of indices where  $\langle x, u_{\alpha_j} \rangle \neq 0$ . Show the partial sums of  $\sum_{j \in \mathbb{N}} \langle x, u_{\alpha_j} \rangle u_{\alpha_j}$  are Cauchy using Pythagorean theorem and so show converges.
- Set

$$y = x - \sum_{j \in \mathbb{N}} \langle x, u_{\alpha_j} \rangle u_{\alpha_j}$$

and show  $\langle y, u_\alpha \rangle = 0$ .

- (ii)  $\implies$  (iii): let  $\varepsilon > 0$ . Use definition of unconditional convergence of  $x$  and Pythagorean theorem to show  $\|x\|^2 - \sum_{\alpha \in I} |\langle x, u_\alpha \rangle|^2 < \varepsilon$ .

□

**Definition.** Orthonormal subset  $U = \{u_\alpha : \alpha \in I\}$  of Hilbert space  $H$  is **complete** if it satisfies any of the conditions in Theorem 7.2.16. An **orthonormal basis** of  $H$  is a complete orthonormal subset of  $H$ .

**Definition.**  $U$  is **maximal orthonormal set** if  $\forall V \subseteq H$  such that  $U \subsetneq V$ ,  $V$  is not orthonormal.

**Lemma.**  $U$  is maximal orthonormal set iff it is an orthonormal basis.

**Remark.** For orthonormal basis  $\{u_\alpha : \alpha \in \mathbb{N}\}$ , representation  $x = \sum_{\alpha \in \mathbb{N}} c_\alpha u_\alpha$  is unique (consider  $\langle x - x, u_\beta \rangle = \lim_{n \rightarrow \infty} \langle \sum_{\alpha=1}^n (c_\alpha - d_\alpha) u_\alpha, u_\beta \rangle$ ).

**Theorem.** Every Hilbert space  $H$  has orthonormal basis. If  $V \subseteq H$  is orthonormal set, then  $H$  has orthonormal basis containing  $V$ .

*Proof.*

- Assume  $H \neq \{0\}$ . Use partial ordering  $\subseteq$ .
- Let  $\{U_\alpha : \alpha \in I\}$  be totally ordered collection of orthonormal sets. Find upper bound of  $\{U_\alpha : \alpha \in I\}$  which is orthonormal.
- Show result using Theorem 7.2.25 and Lemma 7.2.19.
- To show orthonormal sets  $V$  can be extended to orthonormal bases, use same argument on family of all orthonormal subsets of  $H$  containing  $V$ .

□

**Definition.** A set  $X$  is **partially ordered** if it is equipped with relation  $\leq$  satisfying:

- **Reflexivity:**  $\forall x \in X, x \leq x$ .
- **Transitivity:**  $(x \leq y \wedge y \leq z) \implies x \leq z$ .
- **Anti-symmetry:**  $(x \leq y \wedge y \leq x) \implies x = y$ .

$X$  is **totally ordered** if partially ordered and  $\forall x, y \in X$ , either  $x \leq y$  or  $y \leq x$ .

**Definition.** Let  $X$  totally ordered set with relation  $\leq$ .  $x \in X$  is **upper bound** for  $Y \subseteq X$  if  $\forall y \in Y, y \leq x$ .  $x \in X$  is **maximal** if  $\forall y \in X, x \leq y \implies y = x$ .

**Example.** Let  $X$  be non-empty collection of sets. Then  $\subseteq$  is partial ordering on  $X$ .  $A \in X$  is upper bound for  $X' \subseteq X$  if every set in  $X'$  is subset of  $A$ .  $M \in X$  is maximal if it is not proper subset of any set in  $X$ .

**Theorem (Zorn's Lemma).** A partially ordered set  $X$  that has upper bounds for its totally ordered subsets has a maximal element.

**Proposition.** Hilbert space is separable iff it has countable orthonormal basis.

*Proof.*

- $\implies$ : let  $U = \{u_n : n \in \mathbb{N}\}$  countable, dense in  $H$ . Recursively discard any  $u_n$  in linear span of  $u_1, \dots, u_{n-1}$  to obtain linearly independent set  $V = \{v_n : n \in \mathbb{N}\}$  whose linear span is dense in  $H$ . Applying Gram-Schmidt, set

$$w_1 = \frac{v_1}{\|v_1\|}, \dots, w_{n+1} = c_{n+1} \left( v_{n+1} - \sum_{k=1}^n \langle w_k, v_{n+1} w_k \rangle \right)$$

where  $c_n \in \mathbb{C}$  chosen so that  $\|w_n\| = 1$ .  $\{w_n : n \in \mathbb{N}\}$  is countable orthonormal basis.

- $\impliedby$ : let  $\{w_n : n \in \mathbb{N}\}$  be orthonormal basis, show that

$$S_m = \left\{ \sum_{k=1}^m c_k w_k : c_k \in \mathbb{Q} + i\mathbb{Q} \right\}$$

is countable and  $\cup_{m \in \mathbb{N}} S_m$  dense in  $H$ . □

**Theorem (Riesz Representation Theorem for Hilbert Spaces).** Let  $H$  Hilbert space with inner product  $\langle \cdot, \cdot \rangle$ ,  $T : H \rightarrow \mathbb{R}$  bounded linear functional. Then

$$\exists! y \in H : \forall x \in H, T(x) = \langle x, y \rangle$$

Note RHS gives bounded linear functional by Cauchy-Schwarz.

*Proof.*

- Existence:
  - Show  $N = \{x \in H : T(x) = 0\}$  is closed subspace of  $H$ , use that  $H = N \oplus N^\perp$ .
  - Assume  $N^\perp$  contains  $v$  with  $\|v\| = 1$ . For  $x \in H$ , define  $u = T(x)v - T(v)x$ .
  - Show that  $\langle u, v \rangle = 0$ , deduce a value for  $y$  from this.
- Uniqueness: straightforward. □

## 8. Convergence of Fourier series

**Note.** We can view  $f : [-\pi, \pi] \rightarrow \mathbb{C}$  as being  $2\pi$ -periodic by extending it on the real line.

**Definition.**  $m$ -th **partial Fourier sum** of  $2\pi$ -periodic integrable function  $f : [-\pi, \pi] \rightarrow \mathbb{C}$  is given by

$$(S_m f)(x) = \sum_{k=-m}^m a_k(f) e^{ikx}$$

where

$$a_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iky} dy$$

are **Fourier coefficients** of  $f$ .

**Definition.** Let  $f, g : [-\pi, \pi] \rightarrow \mathbb{C}$  be  $2\pi$ -periodic integrable functions. **Convolution**  $f * g$  is

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) g(x - y) dy$$

**Proposition.** Let  $f, g, h : [-\pi, \pi] \rightarrow \mathbb{C}$  be  $2\pi$ -periodic integrable functions,  $c \in \mathbb{C}$ . Then  $*$  satisfies:

- **Commutativity:**  $f * g = g * f$ .
- **Distributivity:**  $f * (g + h) = (f * g) + (f * h)$ .
- **Homogeneity:**  $(cf) * g = c(f * g) = f * (cg)$ .
- **Associativity:**  $(f * g) * h = f * (g * h)$ .

### 8.1. Pointwise convergence of Fourier series via Dirichlet kernel

**Definition.** Let  $m \in \mathbb{N}_0$ . The  $m$ -th **Dirichlet kernel** is

$$D_m(x) := \sum_{k=-m}^m e^{ikx}$$

**Proposition.**

- $D_m$  is trigonometric polynomial of degree  $m$  with coefficients equal to 1 for  $k \in [-m, m]$  and 0 otherwise.
- $D_m$  is real-valued and  $2\pi$ -periodic.
- 

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} D_m(x) dx = 1$$

**Proposition.** Let  $f : [-\pi, \pi] \rightarrow \mathbb{C}$  be  $2\pi$ -periodic integrable function. Then

$$(D_m * f)(x) = \sum_{k=-m}^m a_k(f) e^{ikx} = (S_m f)(x)$$

where  $a_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iky} dy$ .

**Proposition.**

$$D_m(x) = \frac{\sin\left(\left(m + \frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)}$$

**Remark.** RHS in Proposition 8.1.4 has removable singularity at  $x = 0$ , and  $D_m(0) = 2m + 1$ . Applying l'Hopital's rule to RHS gives

$$\lim_{x \rightarrow 0} \frac{\sin\left(\left(m + \frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)} = 2m + 1$$

**Theorem** (Riemann-Lebesgue Lemma). Let  $E \subseteq \mathbb{R}$  measurable,  $f \in L^1(E)$ . Then

$$\lim_{n \rightarrow \infty} \int_E f(x) \sin(nx) = \lim_{n \rightarrow \infty} \int_E f(x) \cos(nx) = \lim_{n \rightarrow \infty} \int_E f(x) e^{-inx} = 0$$

*Proof.*

- First consider when  $f(x) = \mathbf{1}_{(a,b)}(x)$ . Define  $I_j = \left(\frac{2\pi j}{n}, \frac{2\pi(j+1)}{n}\right)$ , so integral of  $\sin(nx)$  over each  $I_j$  is 0.
- Write

$$(a, b) = L \cup \bigcup_{j=1}^N I_j \cup R$$

so that  $\text{length}(L), \text{length}(R) < \frac{2\pi}{n}$ .

- Show that

$$\left| \int_E f(x) \sin(nx) \right| < \frac{4\pi}{n}$$

- Deduce the sin result for step functions.
- Use that step functions are dense in  $L^1$  to show sin result for  $f \in L^1(E)$  by writing  $f = (f - \psi) + \psi$  and finally take lim sup.
- Same argument works for cos.
- Conclude exp result.

□

**Theorem.** Let  $f \in L^1([-\pi, \pi])$  be  $2\pi$ -periodic, assume  $f$  differentiable at  $b \in [-\pi, \pi]$ . Then

$$f(b) = \lim_{m \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) D_m(b-y) dy = \lim_{m \rightarrow \infty} (f * D_m)(b) = \lim_{m \rightarrow \infty} S_m f(b)$$

*Proof.*

- First assume  $b = 0$ . Let  $0 < \varepsilon < 1$ , show that  $f(y)/\sin(y/2)$  is integrable on  $[\varepsilon, \pi]$  and show

$$\lim_{m \rightarrow \infty} \int_{\varepsilon}^{\pi} \frac{f(y)}{\sin\left(\frac{y}{2}\right)} \sin\left(\left(m + \frac{1}{2}\right)y\right) dy = 0$$

Conclude the same for  $\int_{-\pi}^{-\varepsilon}$ .

- Write  $f(y) = f(0) + s(y)$  and split the integral  $\int_{-\pi}^{\pi}$  as such.
- Use Proposition 8.1.2 and split integral of  $s(y)$  to show

$$\lim_{m \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) D_m(y) dy = f(0) + \lim_{m \rightarrow \infty} \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} s(y) D_m(y) dy$$

- Use differentiability at 0 to show for  $\varepsilon$  small and  $y \in [-\varepsilon, \varepsilon]$ ,  $|s(y)| \leq C|y|$ .
- Show that  $|x|/|\sin(x)| \leq 2$  for  $x$  small (for  $\cos(x) \geq \frac{1}{2}$ ) by considering  $g(x) = 2\sin(x) - x$ , and then that

$$0 \leq \left| \lim_{m \rightarrow \infty} \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} s(y) D_m(y) dy \right| \leq \frac{4C\varepsilon}{\pi}$$

- Conclude the result for  $b = 0$ .
- To show for  $b \in [-\pi, \pi]$ , define  $G(y) = f(b - y)$  and use commutativity of convolution.

□

## 8.2. Uniform convergence of Cesàro mean Fourier series via Fejér kernel

**Definition.** Let  $x \in \mathbb{R}$ ,  $N \in \mathbb{N}$ . **Fejér kernel** is

$$F_N(x) = \frac{1}{N} \sum_{m=0}^{N-1} D_m(x) = \frac{1}{N} \sum_{m=0}^{N-1} \sum_{k=-m}^m e^{ikx}$$

**Proposition.**

- $$\frac{1}{2\pi} \int_{-\pi}^{\pi} F_N(x) dx = 1$$

- $$F_N(x) = \frac{1}{N} \left( \frac{\sin(Nx/2)}{\sin(x/2)} \right)^2$$

- Fejér kernel is non-negative, so

$$F_N(x) = |F_N(x)| \implies \int_{-\pi}^{\pi} |F_N(x)| dx = 2\pi$$

- For  $\varepsilon > 0$  and  $\varepsilon < |x| < \pi$ , there exists  $C_\varepsilon > 0$  such that  $(\sin(x/2))^{-2} \leq C_\varepsilon$ , hence

$$\int_{\varepsilon}^{\pi} |F_N(x)| dx = \frac{1}{N} \int_{\varepsilon}^{\pi} \left| \frac{\sin(Nx/2)}{\sin(x/2)} \right|^2 dx \leq \frac{1}{N} \int C_\varepsilon \leq \frac{\pi C_\varepsilon}{N} \rightarrow 0 \quad \text{as } N \rightarrow \infty$$

and similarly for  $-\pi < x < -\varepsilon$ .

**Definition.** The  $N$ -th **Cesàro mean** is the average of the first  $N$  partial Fourier sums of  $f$ :

$$\frac{1}{N} \sum_{m=0}^{N-1} (S_m f)(x)$$



**Proposition.** Let  $f : [-\pi, \pi] \rightarrow \mathbb{C}$  integrable, then convolution of  $f$  with Fejér kernel is the Cesàro mean:

$$(f * F_N)(x) = \frac{1}{N} \sum_{m=0}^{N-1} (S_m f)(x)$$

**Theorem.** Let  $f : [-\pi, \pi] \rightarrow \mathbb{C}$  continuous and  $2\pi$ -periodic, then

$$\forall x \in [-\pi, \pi], \quad f(x) = \lim_{N \rightarrow \infty} (f * F_N)(x) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{m=0}^{N-1} (S_m f)(x)$$

and the convergence is uniform.

*Proof.*

- Reason that  $f$  is bounded:  $|f| \leq B$  on  $[-\pi, \pi]$ .
- Let  $\rho > 0$ . Show that  $\forall x, y \in [-\pi, \pi]$ , for some  $\varepsilon > 0$ ,  $|y| < \varepsilon \implies |f(x - y) - f(x)| < \rho$ .
- Show that

$$\begin{aligned} & |(f * F_N)(x) - f(x)| \\ & \leq \frac{1}{2\pi} \left( \int_{-\pi}^{-\varepsilon} + \int_{\varepsilon}^{\pi} \right) |F_N(y)| |f(x - y) - f(x)| dy + \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} |F_N(y)| |f(x - y) - f(x)| dy \end{aligned}$$

- Show that first terms of RHS tend to zero as  $N \rightarrow \infty$ .
- Show last term on RHS is  $< \rho$ .
- Conclude the result.

□

**Remark.**

- By above theorem, any  $2\pi$ -periodic continuous function on  $[-\pi, \pi]$  can be uniformly approximated by trigonometric polynomials, i.e. if  $\varepsilon > 0$ , then there exists trigonometric polynomial  $p$  such that  $\forall x \in [-\pi, \pi], |f(x) - p(x)| < \varepsilon$ .
- This is analogue of Weierstrass Approximation Theorem for  $2\pi$ -periodic functions. Weierstrass Approximation Theorem states that for continuous function  $f : [a, b] \rightarrow \mathbb{R}$  and  $\varepsilon > 0$ , there exists polynomial  $p$  such that  $\forall x \in [a, b], |f(x) - p(x)| < \varepsilon$ .
- Continuous functions are dense in  $L^p([a, b])$  for  $p \in [1, \infty)$ . Let  $\varepsilon > 0$ ,  $f \in L^p([a, b])$  and  $g : [a, b] \rightarrow \mathbb{R}$  continuous such that  $\|f - g\|_{L^p} < \varepsilon$ . By Weierstrass Approximation Theorem, there exists polynomial  $\tilde{p}$  such that

$$\forall x \in [a, b], \quad |g(x) - \tilde{p}(x)| < \frac{\varepsilon}{(b - a)^{1/p}}$$

Hence

$$\int_a^b |g(x) - \tilde{p}(x)|^p < \varepsilon^p \quad \text{i.e.} \quad \|g - \tilde{p}\|_{L^p} < \varepsilon$$

Hence by Minkowski's inequality,  $\|f - \tilde{p}\|_{L^p} < 2\varepsilon$ . Hence polynomials are dense in  $L^p([a, b])$  for  $p \in [1, \infty)$ .

- **Note:** for  $p = \infty$ , any continuous function in  $L^\infty([a, b])$  can be approximated by polynomials, but continuous functions are not dense in  $L^\infty([a, b])$ .
- Similarly, trigonometric polynomials are dense in  $L^p([-\pi, \pi])$  for  $p \in [1, \infty)$ .

### 8.3. Mean convergence of Fourier series in $L^2([-\pi, \pi])$

**Notation.** Define an inner product on  $L^2([-\pi, \pi])$  by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{[-\pi, \pi]} f \bar{g}$$

and denote  $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ .  $(L^2([-\pi, \pi]), \langle \cdot, \cdot \rangle)$  is Hilbert space by Riesz-Fischer.

For  $k \in \mathbb{Z}$ ,  $x \in [-\pi, \pi]$ , let  $\varphi_k(x) = e^{ikx}$ , then for  $2\pi$ -periodic integrable function  $f : [-\pi, \pi] \rightarrow \mathbb{C}$ ,

$$a_k(f) = \langle f, \varphi_k \rangle, \quad S_N f(x) = \sum_{k=-N}^N \langle f, \varphi_k \rangle \varphi_k$$

**Lemma.** Let  $f \in L^2([-\pi, \pi])$  be  $2\pi$ -periodic, define

$$\mathcal{P}_N = \left\{ \sum_{k=-n}^n c_k \varphi_k : c_k \in \mathbb{C}, n \leq N \right\}$$

Then:

- $\{\varphi_n : n \in \mathbb{Z}\}$  is orthonormal in  $L^2([-\pi, \pi])$  with respect to  $\langle \cdot, \cdot \rangle$ .
- $\forall p \in \mathcal{P}_N$ ,  $f - S_N f$  is orthogonal to  $p$ .
- $\forall N \geq 0$ ,  $\forall p \in \mathcal{P}_N$ ,

$$\|f - S_N f\| \leq \|f - p\|$$

with equality iff  $p = S_N f$ .

*Proof.*

- Show  $\frac{1}{2\pi} \int_{[-\pi, \pi]} \varphi_m \bar{\varphi}_n = 0 = \delta_{mn}$  (justify use of Riemann integral).
- Show that  $(f - S_N f) \perp \varphi_m$  for each  $|m| \leq N$  to show  $(f - S_N f) \perp p$  for  $p \in \mathcal{P}_N$ .
- Write  $f - p = f - S_N f + S_N f - \sum_{k=-N}^N c_k \varphi_k$ , use Pythagoras.

□

**Remark.** Above lemma is projection result, i.e.  $S_N f$  is best approximation to  $f$  in  $\mathcal{P}_N$ .

**Theorem.** Let  $f \in L^2([-\pi, \pi])$  be  $2\pi$ -periodic function. Then Fourier series for  $f$  converges to  $f$  in  $(L^2([-\pi, \pi]), \|\cdot\|)$ , i.e.

$$\lim_{N \rightarrow \infty} \|S_N f - f\| = 0$$

*Proof.*

- First show if  $g : [-\pi, \pi] \rightarrow \mathbb{C}$  continuous, then  $\|S_N g - G\| \rightarrow 0$  as  $N \rightarrow \infty$ .
  - Let  $\varepsilon > 0$ , then for some  $M$ , there exists  $p \in \mathcal{P}_M$  such that
 
$$\forall x \in [-\pi, \pi], \quad |g(x) - p(x)| < \varepsilon$$
  - Use that  $g(x) = \lim_{N \rightarrow \infty} (g * F_N)(x)$  and  $g * F_{M+1} \in \mathcal{P}_M$ .
  - Deduce that  $\|g - p\|^2 < \varepsilon^2$ .
  - Show if  $M \leq N$  then  $\|g - S_N g\| \leq \|g - p\| < \varepsilon$ , conclude result for continuous functions.
- Let  $f \in L^2([-\pi, \pi])$ ,  $\varepsilon > 0$ . Using that continuous functions are dense in  $L^2([-\pi, \pi])$ , there is  $g : [-\pi, \pi] \rightarrow \mathbb{C}$  such that  $\|f - g\| < \varepsilon$ .
- Since  $g$  continuous, for large enough  $M$ ,  $\|S_M g - g\| < \varepsilon$  by above.
- Use triangle inequality, the fact that  $N \geq M \implies S_M g \in \mathcal{P}_N$  and projection theorem to conclude the result.

□

**Lemma.**  $\{\varphi_n : n \in \mathbb{Z}\}$  is orthonormal basis of  $(L^2([-\pi, \pi]))$  with respect to inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{[-\pi, \pi]} f \bar{g}$$

*Proof.*

- Note that  $(L^2([-\pi, \pi]), \langle \cdot, \cdot \rangle)$  is Hilbert space.
- Show Parseval's identity holds.
- Write  $f = f - S_N f + S_N f$ , use projection theorem, Pythagorean theorem and orthonormality of  $\{\varphi_n : n \in \mathbb{Z}\}$  to show

$$\|f\|^2 = \|f - S_N f\|^2 + \sum_{k=-N}^N |\langle f, \varphi_k \rangle|^2$$

- Take limit as  $N \rightarrow \infty$  to conclude result.

□