

Contents

1. Introduction	2
1.1. Cubic equations over \mathbb{C}	2
1.2. Quartic equations over \mathbb{C}	2
2. Fields and polynomials	3
2.1. Basic properties of fields	3
2.2. Polynomials over fields	4
2.3. Tests for irreducibility	4
3. Field extensions	5
3.1. Definitions and examples	5
3.2. Algebraic elements and minimal polynomials	6
3.3. Constructing field extensions	6
3.4. Explicit examples of simple extensions	7
3.5. Degrees of field extensions	7
4. Galois extensions	8
4.1. Splitting fields	8
4.2. Normal extensions	9
4.3. Separable extensions	11
4.4. The fundamental theorem of Galois theory	12
4.5. Computations with Galois groups	13
5. Cyclotomic field extensions	16
5.1. Roots of unity	16
5.2. n -th cyclotomic field extensions	16
5.3. Galois properties of cyclotomic extensions	17
5.4. Special properties of $\mathbb{Q}(\zeta_p)$, where $p > 2$ is prime	17
6. Cyclic field extensions	18
6.1. Cyclic extensions of degree 2	18
6.2. Cyclic extensions of degree n (the Kummer theory)	19
7. Finite fields	19
7.1. Existence and uniqueness	19
7.2. Counting irreducible polynomials over finite fields	20
8. Galois groups of polynomials	20
8.1. Symmetric functions	20
8.2. Galois theory for cubic polynomials	22
8.3. Galois theory for quartic polynomials	24
8.4. A criterion for solvability by radicals	28
8.5. Polynomials not solvable by radicals	29

1. Introduction

Definition. **Epimorphism** is surjective homomorphism.

Definition. **Embedding** or **monomorphism** is injective homomorphism.

1.1. Cubic equations over \mathbb{C}

- For a polynomial equation, a **solution by radicals** is a formula for solutions using only addition, subtraction, multiplication, division and radicals $\sqrt[m]{\cdot}$ for $m \in \mathbb{N}$.
- For general cubic equation $x^3 + a_2x^2 + a_1x + a_0 = 0$:
 - ▶ **Tschirnhaus transformation** is substitution $t = x + \frac{a_2}{3}$, giving

$$t^3 + pt + q = 0, \quad p := \frac{-a_2^2 + 3a_1}{3}, \quad q := \frac{2a_2^3 - 9a_1a_2 + 27a_0}{27}$$

This is a **reduced** (or **depressed**) cubic equation.

- ▶ When $t = u + v$, $t^3 - (3uv)t - (u^3 + v^3) = 0$ which is in the reduced cubic form with $p = -3uv, q = -(u^3 + v^3)$.
- ▶ We have

$$(y - u^3)(y - v^3) = y^2 - (u^3 + v^3)y + u^3v^3 = y^2 + qy - \frac{p^3}{27} = 0$$

so $u^3, v^3 = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$.

- ▶ So a solution to $t^3 + pt + q = 0$ is

$$t = u + v = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

The other solutions are $\omega u + \omega^2 v$ and $\omega^2 u + \omega v$ where $\omega = e^{2\pi i/3}$ is the 3rd root of unity. This is because u and v each have three solutions independently to $u^3, v^3 = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$, but also $uv = -\frac{p}{3}$.

Remark. The above method doesn't work for fields of characteristic 2 or 3 since the formulas involve division by 2 or 3 (which is dividing by zero in these respective fields).

1.2. Quartic equations over \mathbb{C}

- For general quartic equation $x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$:
 - ▶ Substitution $t = x + \frac{a_3}{4}$ gives **reduced** quartic equation

$$t^4 + pt^2 + qt + r = 0$$

- ▶ We then manipulate the polynomial so that it is the sum or difference of two squares and use $a^2 + b^2 = (a + ib)(a - ib)$ or $a^2 - b^2 = (a + b)(a - b)$:

$$(t^2 + w)^2 + (p - 2w)t^2 + qt + (r - w^2) = 0$$

- ▶ $(p - 2w)t^2 + qt + (r - w^2) = 0$ is a square iff its discriminant is zero:

$$q^2 - 4(p - 2w)(r - w^2) = 0 \iff w^3 - \frac{1}{2}pw^2 - rw + \frac{1}{8}(4pr - q^2) = 0$$

- This **cubic resolvent** is solvable by radicals. Taking any of the solutions and substituting for w gives a sum or difference of two squares in t . The quadratic factors can then be solved.

2. Fields and polynomials

2.1. Basic properties of fields

Definition. Ring R is **field** if every element of $R - \{0\}$ has multiplicative inverse and $1 \neq 0 \in R$.

Lemma. Every field is integral domain.

Definition. **Field homomorphism** is ring homomorphism $\varphi : K \rightarrow L$ between fields:

- $\varphi(a + b) = \varphi(a) + \varphi(b)$
- $\varphi(ab) = \varphi(a)\varphi(b)$
- $\varphi(1) = 1$

These imply $\varphi(0) = 0$, $\varphi(-a) = -\varphi(a)$, $\varphi(a^{-1}) = \varphi(a)^{-1}$.

Lemma. Let $\varphi : K \rightarrow L$ field homomorphism.

- $\text{im}(\varphi) = \{\varphi(a) : a \in K\}$ is field.
- $\text{ker}(\varphi) = \{a \in K : \varphi(a) = 0\} = \{0\}$, i.e. φ is injective.

Definition. **Subfield** K of field L is subring of L where K is field. L is **field extension** of K .

- The above lemma shows image of $\varphi : K \rightarrow L$ is subfield of L .

Lemma. Intersections of subfields are subfields.

Definition. **Prime subfield** of L is intersection of all subfields of L .

Definition. **Characteristic** $\text{char}(K)$ of field K is

$$\text{char}(K) := \min\{n \in \mathbb{N} : \chi(n) = 0\}$$

(or 0 if this does not exist) where $\chi : \mathbb{Z} \rightarrow K$, $\chi(m) = 1 + \dots + 1$ (m times).

Example. $\text{char}(\mathbb{Q}) = \text{char}(\mathbb{R}) = \text{char}(\mathbb{C}) = 0$, $\text{char}(\mathbb{F}_p) = p$ for p prime.

Lemma. For any field K , $\text{char}(K)$ is either 0 or prime.

Theorem.

- If $\text{char}(K) = 0$ then prime subfield of K is $\cong \mathbb{Q}$.
- If $\text{char}(K) = p > 0$ then prime subfield of K is $\cong \mathbb{F}_p$.

Corollary.

- If \mathbb{Q} is subfield of K then $\text{char}(K) = 0$.
- If \mathbb{F}_p is subfield of K for prime p then $\text{char}(K) = p$.

Remark. Let $\text{char}(K) = p$, then $p \mid \binom{p}{i}$ so $(a + b)^p = a^p + b^p$ in K . Also in $K[x]$ for p prime, $x^p - 1 = (x - 1)^p$.

Theorem (Fermat's little theorem). $\forall a \in \mathbb{F}_p, a^p = a$.

2.2. Polynomials over fields

Definition. Degree of $f(x) = a_0 + a_1x + \dots + a_nx^n, a_n \neq 0$, is $\deg(f(x)) = n$.

- Degree of zero polynomial is $\deg(0) = -\infty$.
- $\deg(f(x)g(x)) = \deg(f(x)) + \deg(g(x))$.
- $\deg(f(x) + g(x)) \leq \max\{\deg(f(x)), \deg(g(x))\}$ with equality if $\deg(f(x)) \neq \deg(g(x))$.
- Only invertible elements in $K[x]$ are non-zero constants $f(x) = a_0 \neq 0$.
- Similarities between \mathbb{Z} and $K[x]$ for field K :
 - $K[x]$ is integral domain.
 - There is a division algorithm for $K[x]$: for $f(x), g(x) \in K[x], \exists!q(x), r(x) \in K[x]$ with $\deg(r(x)) < \deg(g(x))$ such that

$$f(x) = q(x)g(x) + r(x)$$

- Every $f(x), g(x) \in K[x]$ have greatest common divisor $\gcd(f(x), g(x))$ unique up to multiplication by non-zero constants. By Euclidean algorithm for polynomials,

$$\exists a(x), b(x) \in K[x] : a(x)f(x) + b(x)g(x) = \gcd(f(x), g(x))$$

- Can construct field from $K[x]$: **field of fractions** of $K[x]$ is

$$K(x) := \text{Frac}(K[x]) = \left\{ \frac{f(x)}{g(x)} : f(x), g(x) \in K[x], g(x) \neq 0 \right\}$$

where $f_1(x)/g_1(x) = f_2(x)/g_2(x) \iff f_1(x)g_2(x) = f_2(x)g_1(x)$. (We can construct the field of fractions for any integral domain).

- $K[x]$ is PID and so UFD.

Definition. For field $K, f(x) \in K[x]$ **irreducible in $K[x]$** (or $f(x)$ is **irreducible over K**) if

- $\deg(f(x)) \geq 1$ and
- $f(x) = g(x)h(x) \implies g(x)$ or $h(x)$ is constant

2.3. Tests for irreducibility

Proposition. If $f(x)$ has linear factor in $K[x]$, it has root in $K[x]$.

Proposition (Rational root test). If $f(x) = a_0 + \dots + a_nx^n \in \mathbb{Z}[x]$ has rational root $\frac{b}{c} \in \mathbb{Q}$ with $\gcd(b, c) = 1$ then $b \mid a_0$ and $c \mid a_n$. **Note:** this can't be used to show f is irreducible for $\deg(f(x)) \geq 4$.

Theorem (Gauss's lemma). Let $f(x) \in \mathbb{Z}[x], f(x) = g(x)h(x), g(x), h(x) \in \mathbb{Q}[x]$. Then $\exists r \in \mathbb{Q} : rg(x), r^{-1}h(x) \in \mathbb{Z}[x]$. i.e. if $f(x)$ can be factored in $\mathbb{Q}[x]$ it can be factored in $\mathbb{Z}[x]$.

Example. Let $f(x) = x^4 - 3x^3 + 1 \in \mathbb{Q}[x]$. Using the rational root test, $f(\pm 1) \neq 0$ so no linear factors in $\mathbb{Q}[x]$. Checking quadratic factors, let

$$f(x) = (ax^2 + bx + c)(rx^2 + sx + t), \quad a, b, c, r, s, t \in \mathbb{Z} \text{ by Gauss's lemma}$$

So $1 = ar \Rightarrow a = r = \pm 1$. $1 = ct \Rightarrow c = t = \pm 1$. $-3 = b + s$ and $0 = c(b + s)$: contradiction. So $f(x)$ irreducible in $\mathbb{Q}[x]$.

Example. Let $f(x) = x^4 - 3x^2 + 1 \in \mathbb{Q}[x]$. The rational root test shows there are no linear factors. Checking quadratic factors, let

$$f(x) = (ax^2 + bx + c)(rx^2 + sx + t), \quad a, b, c, r, s, t \in \mathbb{Z} \text{ by Gauss's lemma}$$

As before, $a = r = \pm 1$, $c = t = \pm 1$. $0 = b + s \Rightarrow b = -s$, $-3 = at + bs + cr = -b^2 \pm 2$. $b = 1$ works. So $f(x) = (x^2 - x - 1)(x^2 + x - 1)$.

Proposition. Let $f(x) = a_0 + \dots + a_n x^n \in \mathbb{Z}[x]$. If exists prime $p \nmid a_n$ such that $\bar{f}(x)$ is irreducible in $\mathbb{F}_p[x]$, then $f(x)$ irreducible in $\mathbb{Q}[x]$.

Example. Let $f(x) = 8x^3 + 14x - 9$. Reducing mod 7, $\bar{f}(x) = x^3 - 2 \in \mathbb{F}_7[x]$. No roots exist for this, so $f(x)$ irreducible in $\mathbb{Q}[x]$. For some polynomials, no p is suitable, e.g. $f(x) = x^4 + 1$.

- Gauss's lemma works with any UFD R instead of \mathbb{Z} and field of fractions $\text{Frac}(R)$ instead of \mathbb{Q} : e.g. let F field, $R = F[t]$, $K = F(t)$, then $f(x) \in R[x]$ irreducible in $K[x]$ if $f(x)$ is irreducible in $R[x]$.

Proposition (Eisenstein's criterion). Let $f(x) = a_0 + \dots + a_n x^n \in \mathbb{Z}[x]$, prime $p \in \mathbb{Z}$ such that $p \mid a_0, \dots, p \mid a_{n-1}$, $p \nmid a_n$, $p^2 \nmid a_0$. Then $f(x)$ irreducible in $\mathbb{Q}[x]$.

Example. Let $f(x) = x^3 - 3x + 1$. Consider $f(x - 1) = x^3 - 3x^2 + 3$. Then by Eisenstein's criterion with $p = 3$, $f(x - 1)$ irreducible in $\mathbb{Q}[x]$ so $f(x)$ is as well, since factoring $f(x - 1)$ is equivalent to factoring $f(x)$.

Example. p -th cyclotomic polynomial is

$$f(x) = \frac{x^p - 1}{x - 1} = 1 + \dots + x^{p-1}$$

Now

$$f(x + 1) = \frac{(1 + x)^p - 1}{1 + x - 1} = x^{p-1} + px^{p-2} + \dots + \binom{p}{p-2}x + p$$

so can apply Eisenstein with $p = p$.

Proposition (Generalised Eisenstein's criterion). Let R be integral domain, $K = \text{Frac}(R)$,

$$f(x) = a_0 + \dots + a_n x^n \in R[x]$$

If there is irreducible $p \in R$ with

$$p \mid a_0, \dots, p \mid a_{n-1}, p \nmid a_n, p^2 \nmid a_0$$

then $f(x)$ is irreducible in $K[x]$.

3. Field extensions

3.1. Definitions and examples

Definition. Field extension L/K is field L containing subfield K . Can specify homomorphism $\iota : K \rightarrow L$ (which is injective).

Example.

- $\mathbb{C}/\mathbb{R}, \mathbb{C}/\mathbb{Q}, \mathbb{R}/\mathbb{Q}$.
- $L = \mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is field extension of \mathbb{Q} . $\mathbb{Q}(\theta)$ is field extension of \mathbb{Q} where θ is root of $f(x) \in \mathbb{Q}[x]$.
- $L = \mathbb{Q}(\sqrt[3]{2}) = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} : a, b, c \in \mathbb{Q}\}$ is smallest subfield of \mathbb{R} containing \mathbb{Q} and $\sqrt[3]{2}$.
- $K(t)$ is field extension of K .

Definition. Let L/K field extension, $S \subseteq L$. Then K **with S adjoined**, $K(S)$, is minimal subfield of L containing K and S . If $|S| = 1$, L/K is a **simple extension**.

Example. $\mathbb{Q}(\sqrt{2}, \sqrt{7}) = \{a + b\sqrt{2} + c\sqrt{7} + d\sqrt{14} : a, b, c, d \in \mathbb{Q}\}$ is \mathbb{Q} with $S = \{\sqrt{2}, \sqrt{7}\}$.

Example. \mathbb{R}/\mathbb{Q} is not simple extension.

Definition. Tower is chain of field extensions, e.g. $K \subset M \subset L$.

3.2. Algebraic elements and minimal polynomials

Definition. Let L/K field extension, $\theta \in L$. Then θ is **algebraic over K** if

$$\exists 0 \neq f(x) \in K[x] : f(\theta) = 0$$

Otherwise, θ is **transcendental over K** .

Example. For $n \geq 1$, $\theta = e^{2\pi i/n}$ is algebraic over \mathbb{Q} (root of $x^n - 1$).

Example. $t \in K(t)$ is transcendental over K .

Lemma. The algebraic elements in $K(t)/K$ are precisely K .

Lemma. Let L/K field extension, $\theta \in L$. Define $I_K(\theta) := \{f(x) \in K[x] : f(\theta) = 0\}$. Then $I_K(\theta)$ is ideal in $K[x]$ and

- If θ transcendental over K , $I_K(\theta) = \{0\}$
- If θ algebraic over K , then exists unique monic irreducible polynomial $m(x) \in K[x]$ such that $I_K(\theta) = \langle m(x) \rangle$.

Definition. For $\theta \in L$ algebraic over K , **minimal polynomial of θ over K** is the unique monic polynomial $m(x) \in K[x]$ such that $I_K(\theta) = \langle m(x) \rangle$. The **degree** of θ over K is $\deg(m(x))$.

Remark. If $f(x) \in K[x]$ irreducible over K , monic and $f(\theta) = 0$ then $f(x) = m(x)$.

Example.

- Any $\theta \in K$ has minimal polynomial $x - \theta$ over K .
- $i \in \mathbb{C}$ has minimal polynomial $x^2 + 1$ over \mathbb{R} .
- $\sqrt{2}$ has minimal polynomial $x^2 - 2$ over \mathbb{Q} . $\sqrt[3]{2}$ has minimal polynomial $x^3 - 2$ over \mathbb{Q} .

3.3. Constructing field extensions

Lemma. Let K field, $f(x) \in K[x]$ non-zero. Then

$f(x)$ irreducible over $K \iff K[x]/\langle f(x) \rangle$ is a field

Definition. Let $L_1/K, L_2/K$ field extensions, $\varphi : L_1 \rightarrow L_2$ field homomorphism. φ is **K -homomorphism** if $\forall a \in K, \varphi(a) = a$ (φ fixes elements of K).

- If φ is isomorphism then it is **K -isomorphism**.
- If $L_1 = L_2$ and φ is bijective then φ is **K -automorphism**.

Theorem. Let $m(x) \in K[x]$ irreducible, monic, $K_m := K[x]/\langle m(x) \rangle$. Then

- K_m/K is field extension.
- Let $\theta = \pi(x)$ where $\pi : K[x] \rightarrow K_m$ is canonical projection, then θ has minimal polynomial $m(x)$ and $K_m \cong K(\theta)$.

Proposition (Universal property of simple extension). Let L/K field extension, $\tau \in L$ with $m(\tau) = 0$ and $K_L(\tau)$ be minimal subfield of L containing K and τ . Then exists unique K -isomorphism $\varphi : K_m \rightarrow K_L(\tau)$ such that $\varphi(\theta) = \tau$.

Example.

- Complex conjugation $\mathbb{C} \rightarrow \mathbb{C}$ is \mathbb{R} -automorphism.
- Let K field, $\text{char}(K) \neq 2, \sqrt{2} \notin K$, so $x^2 - 2$ is minimal polynomial of $\sqrt{2}$ over K , then $K(\sqrt{2}) \cong K[x]/\langle x^2 - 2 \rangle$ is field extension of K and $a + b\sqrt{2} \mapsto a - b\sqrt{2}$ is K -automorphism.

Proposition. Let θ transcendental over K , then exists unique K -isomorphism $\varphi : K(t) \rightarrow K(\theta)$ such that $\varphi(t) = \theta$:

$$\varphi\left(\frac{f(t)}{g(t)}\right) = \varphi\left(\frac{f(\theta)}{g(\theta)}\right)$$

3.4. Explicit examples of simple extensions

Example.

- Let $r \in K^\times$ non-square in $K, \text{char}(K) \neq 2$, then $x^2 - r$ irreducible in $K[x]$. E.g. for $K = \mathbb{Q}(t), x^2 - t \in K[x]$ is irreducible. Then $K(\sqrt{t}) = \mathbb{Q}(\sqrt{t}) \cong K[x]/\langle x^2 - t \rangle$.
- Define $\mathbb{F}_9 = \mathbb{F}_3[x]/\langle x^2 - 2 \rangle \cong \mathbb{F}_3(\theta) = \{a + b\theta : a, b \in \mathbb{F}_3\}$ for θ a root of $x^2 - 2$.

Proposition. Let $K(\theta)/K$ where θ has minimal polynomial $m(x) \in K[x]$ of degree n . Then

$$K[x]/\langle m(x) \rangle \cong K(\theta) = \{c_0 + c_1\theta + \dots + c_{n-1}\theta^{n-1} : c_i \in K\}$$

and its elements are written uniquely: $K(\theta)$ is vector space over K of dimension n with basis $\{1, \theta, \dots, \theta^{n-1}\}$.

Example. $\mathbb{Q}(\sqrt[3]{2}) = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} : a, b, c \in \mathbb{Q}\} \cong \mathbb{Q}[x]/\langle x^3 - 2 \rangle$. $\mathbb{Q}(\omega\sqrt[3]{2})$ and $\mathbb{Q}(\omega^2\sqrt[3]{2})$ where $\omega = e^{2\pi i/3}$ are isomorphic to $\mathbb{Q}(\sqrt[3]{2})$ as $\omega\sqrt[3]{2}, \omega\sqrt[3]{4}$ have same minimal polynomial.

3.5. Degrees of field extensions

Definition. Degree of field extension L/K is

$$[L : K] := \dim_K(L)$$

Example.

- When θ algebraic over K of degree n , $[K(\theta) : K] = n$.
- Let θ transcendental over K , then $[K(\theta) : K] = \infty$, so $[K(t) : K] = \infty$, $[\mathbb{Q}(\pi) : \mathbb{Q}]$, $[\mathbb{R} : \mathbb{Q}] = \infty$.

Definition. L/K is **algebraic extension** if every element in L is algebraic over K .

Proposition. Let $[L : K] < \infty$, then L/K is algebraic extension and $L = K(\alpha_1, \dots, \alpha_n)$ for some $\alpha_1, \dots, \alpha_n \in L$. The converse also holds.

Theorem (Tower law). Let $K \subseteq M \subseteq L$ tower of field extensions. Then

- $[L : K] < \infty \iff [L : M] < \infty \wedge [M : K] < \infty$.
- $[L : K] = [L : M][M : K]$.

Example.

- $K = \mathbb{Q} \subset M = \mathbb{Q}(\sqrt{2}) \subset L = \mathbb{Q}(\sqrt{2}, \sqrt{7})$. M/K has basis $\{1, \sqrt{2}\}$ so $[M : K] = 2$. Let $\sqrt{7} \in \mathbb{Q}(\sqrt{2})$, then $\sqrt{7} = c + d\sqrt{2}$, $c, d \in \mathbb{Q}$ so $7 = (c^2 + 2d^2) + 2cd\sqrt{2}$ so $7 = c^2 + 2d^2$, $0 = 2cd$ so $d^2 = \frac{7}{2}$ or $c^2 = 7$, which are both contradictions. So $[L : K] = 4$ with basis $\{1, \sqrt{2}, \sqrt{7}, \sqrt{14}\}$.
- Let $K = \mathbb{Q} \subset M = \mathbb{Q}(i) \subset \mathbb{Q}(i, \sqrt{2})$. We know $[\mathbb{Q}(i) : \mathbb{Q}] = 2$, and $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$, $[\mathbb{Q}(i, \sqrt{2}) : \mathbb{Q}] = 4$ (since $i \notin \mathbb{R}$) so $[\mathbb{Q}(i, \sqrt{2}) : \mathbb{Q}(\sqrt{2})] = 2$.
- Let $K = \mathbb{Q} \subset M = \mathbb{Q}(\sqrt{2}) \subset L = \mathbb{Q}(\sqrt{2}, \sqrt[3]{3})$. Then $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$, $[\mathbb{Q}(\sqrt[3]{3}) : \mathbb{Q}] = 3$ so $2 \mid [L : K]$ and $3 \mid [L : K]$ so $6 \mid [L : K]$ so $[L : K] \geq 6$. But $[L : M] \leq 3$ and $[M : K] \leq 2$ so $[L : K] \leq 6$ hence $[L : K] = 6$.
- More generally, we have $[K(\alpha, \beta) : K] \leq [K(\alpha) : K][K(\beta) : K]$.

Example.

- Let $\theta = \sqrt[3]{4} + 1$. $\mathbb{Q}(\theta) = \mathbb{Q}(\sqrt[3]{4})$ so minimal polynomial over \mathbb{Q} , m , has $\deg(m) = 3$. $(\theta - 1)^3 = 4$ so minimal polynomial is $x^3 - 3x^2 + 3x - 5$.
- Let $\theta = \sqrt{2} + \sqrt{3}$. $\mathbb{Q}(\sqrt{2}, \theta) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ which has degree 2 over $\mathbb{Q}(\sqrt{2})$ so minimal polynomial of θ over $\mathbb{Q}(\sqrt{2})$ has degree 2, $\theta - \sqrt{2} = \sqrt{3}$ so minimal polynomial is $x^2 - 2\sqrt{2}x - 1$.
- Let $\theta = \sqrt{2} + \sqrt{3}$. $\mathbb{Q} \subset \mathbb{Q}(\theta) \subset \mathbb{Q}(\sqrt{2}, \sqrt{3})$ so $[\mathbb{Q}(\theta) : \mathbb{Q}] \mid [\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$ so $[\mathbb{Q}(\theta) : \mathbb{Q}] \in \{1, 2, 4\}$. Can't be 1 as $\theta \notin \mathbb{Q}$. If it was 2 then $1, \theta, \theta^2$ are linearly dependent over \mathbb{Q} which leads to a contradiction. So degree of minimal polynomial of θ over \mathbb{Q} is 4. $\theta^2 = 5 + 2\sqrt{6} \Rightarrow (\theta^2 - 5)^2 = 24$ so minimal polynomial is $x^4 - 10x^2 + 1$.

4. Galois extensions

4.1. Splitting fields

Definition. For field K , $0 \neq f(x) \in K[x]$, L/K is **splitting field** of $f(x)$ over K if

- $\exists c \in K^\times, \theta_1, \dots, \theta_n \in L : f(x) = c(x - \theta_1) \cdots (x - \theta_n)$ ($f(x)$ splits over L).
- $L = K(\theta_1, \dots, \theta_n)$.

Example.

- \mathbb{C} is splitting field of $x^2 + 1$ over \mathbb{R} , since $x^2 + 1 = (x + i)(x - i)$ and $\mathbb{C} = \mathbb{R}(i, -i) = \mathbb{R}(i)$.

- \mathbb{C} is not splitting field of $x^2 + 1$ over \mathbb{Q} as $\mathbb{C} \neq \mathbb{Q}(i, -i)$.
- \mathbb{Q} is splitting field of $x^2 - 36$ over \mathbb{Q} .
- \mathbb{C} is splitting of $x^4 + 1$ over \mathbb{R} .
- $\mathbb{Q}(i, \sqrt{2})$ is splitting field of $x^4 - x^2 - 2 = (x^2 + 1)(x^2 - 2) = (x + i)(x - i)(x + \sqrt{2})(x - \sqrt{2})$ over \mathbb{Q} .
- $\mathbb{F}_2(\theta)$ where $\theta^3 + \theta + 1 = 0$ is splitting field of $x^3 + x + 1$ over \mathbb{F}_2 .
- Consider splitting field of $x^3 - 2$ over \mathbb{Q} . Let $\omega = e^{2\pi i/3} = (-1 + \sqrt{-3})/2$ then $\mathbb{Q}(\sqrt[3]{2}, \omega)$ is splitting field since it must contain $\sqrt[3]{2}, \omega\sqrt[3]{2}, \omega^2\sqrt[3]{2}$.

Theorem. Let $0 \neq f(x) \in K[x]$, $\deg(f) = n$. Then there exists a splitting field L of $f(x)$ over K with

$$[L : K] \leq n!$$

Notation. For field homomorphism $\varphi : K \rightarrow K'$ and $f(x) = a_0 + \dots + a_n x^n \in K[x]$, write

$$\varphi_*(f(x)) := \varphi(a_0) + \dots + \varphi(a_n)x^n \in K'[x]$$

Lemma. Let $\sigma : K \rightarrow K'$ isomorphism and $K(\theta)/K$, θ has minimal polynomial $m(x) \in K[x]$, θ' be root of $\sigma_*(m(x))$. Then there exists unique K -isomorphism $\tau : K(\theta) \rightarrow K'(\theta')$ such that $\tau(\theta) = \theta'$.

Theorem. For field isomorphism $\sigma : K \rightarrow K'$ and $0 \neq f(x) \in K[x]$, let L be splitting field of $f(x)$ over K , L' be splitting field of $\sigma_*(f(x))$ over K' . Then there exists a field isomorphism $\tau : L \rightarrow L'$ such that $\forall a \in K, \tau(a) = \sigma(a)$.

Corollary. Setting $K = K'$ and $\sigma = \text{id}$ implies that splitting fields are unique.

4.2. Normal extensions

Definition. L/K is **normal** if: for all $f(x) \in K[x]$, if f is irreducible and has a root in L then all its roots are in L . In particular, $f(x)$ splits completely as product of linear factors in $L[x]$. So the minimal polynomial of $\theta \in L$ over K has all its roots in L and can be written as product of linear factors in $L[x]$.

Example.

- If $[L : K] = 1$ then L/K is normal.
- If $[L : K] = 2$ then L/K is normal: let $\theta \in L$ have minimal polynomial $m(x) \in K[x]$, then $K \subseteq K(\theta) \subseteq L$ so $\deg(m(x)) = [K(\theta) : K] \in \{1, 2\}$:
 - If $\deg(m(x)) = 1$ then $m(x)$ is already linear.
 - If $\deg(m(x)) = 2$ then $m(x) = (x - \theta)m_1(x)$, $m_1(x) \in L[x]$ is linear so $m(x)$ splits completely in $L[x]$.
- If $[L : K] = 3$ then L/K is not necessarily normal. Let θ be root of $x^3 - 2 \in \mathbb{Q}[x]$. Other two roots are $\omega\theta, \omega^2\theta$ where $\omega = e^{2\pi i/3}$. If $\omega\theta \in \mathbb{Q}(\theta)$ then $\omega = \frac{\omega\theta}{\theta} \in L$ so $\mathbb{Q} \subset \mathbb{Q}(\omega) \subset \mathbb{Q}(\theta)$ but $[\mathbb{Q}(\omega) : \mathbb{Q}] = 2$ which doesn't divide $[\mathbb{Q}(\theta) : \mathbb{Q}] = 3$.
- Let $\theta \in \mathbb{C}$ be root of irreducible $f(x) = x^3 - 3x - 1 \in \mathbb{Q}[x]$. Let $\theta = u + v$, then $(u + v)^3 - 3uv(u + v) - (u^3 + v^3) \equiv 0$ implies $uv = 1 = u^3v^3, u^3 + v^3 = 1$. So $(y - u^3)(y - v^3) = y^2 - y + 1$ has roots u^3 and v^3 . So the three roots of f are

$$\begin{aligned}\theta_1 &= u + v = e^{\pi i/9} + e^{-\pi i/9} = 2 \cos(\pi/9) \\ \theta_2 &= \omega u + \omega^2 v = e^{7\pi i/9} + e^{-7\pi i/9} = 2 \cos(7\pi/9) \\ \theta_3 &= \omega^2 u + \omega v = e^{13\pi i/9} + e^{-13\pi i/9} = 2 \cos(13\pi/9)\end{aligned}$$

Furthermore, for each i, j , $\theta_i \in \mathbb{Q}(\theta_j)$, e.g.

$$\theta_2 = 2 \cos\left(\pi - \frac{2\pi}{9}\right) = -2 \cos\left(\frac{2\pi}{9}\right) = -2 \left(2 \cos\left(\frac{\pi}{9}\right)^2 - 1\right) = 2 - \theta_1^2$$

Also $\theta_1 + \theta_2 + \theta_3 = 0$ so $\theta_3 \in \mathbb{Q}(\theta_1)$. So $\mathbb{Q}(\theta_1)$ contains all roots of $f(x)$.

Theorem (normality criterion). L/K is finite and normal iff L is splitting field for some $0 \neq f(x) \in K[x]$ over K .

Example.

- $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7})/\mathbb{Q}$ is normal as it is the splitting field of $f(x) = (x^2 - 2)(x^2 - 3)(x^2 - 5)(x^2 - 7) \in \mathbb{Q}[x]$.
- $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not normal but $\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}$ is normal as it is the splitting field of $x^3 - 2 \in \mathbb{Q}$.
- $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$ is not normal but $\mathbb{Q}(\sqrt[4]{2}, i)/\mathbb{Q}$ is normal.
- Let θ root of $f(x) = x^3 - 3x - 1 \in \mathbb{Q}[x]$. Then $\mathbb{Q}(\theta)/\mathbb{Q}$ is normal as is splitting field of $f(x)$ over \mathbb{Q} .
- $\mathbb{F}_2(\theta)/\mathbb{F}_2$ where $\theta^3 + \theta^2 + 1 = 0$ is normal, as $\mathbb{F}_2(\theta)$ contains all roots of $x^3 + x^2 + 1$.
- $\mathbb{F}_p(\theta)/\mathbb{F}_p(t)$ where $\theta^p = t$ is normal as it is the splitting field of $x^p - t = x^p - \theta^p = (x - \theta)^p$ so $f(x)$ splits into linear factors in $L[x]$.

Definition. Field N is **normal closure** of L/K if $K \subseteq L \subseteq N$, N/K is normal, and if $K \subseteq L \subseteq N' \subseteq N$ with N'/K normal then $N = N'$.

Theorem. Every finite extension L/K has normal closure, unique up to a K -isomorphism.

Definition. $\text{Aut}(L/K)$ is group of K -automorphisms of L/K with composition as the group operation.

Example.

- $\text{Aut}(\mathbb{C}/\mathbb{R})$ contains at least two elements: complex conjugation: $\sigma(a + bi) = a - bi$ and the identity map $\text{id} = \sigma^2$. If $\tau \in \text{Aut}(\mathbb{C}/\mathbb{R})$ then $\tau(a + bi) = a + b\tau(i)$. But $\tau(i)^2 = \tau(i^2) = \tau(-1) = -1$ hence $\tau(i) = \pm i$. So there are only two choices for τ . So $\text{Aut}(\mathbb{C}/\mathbb{R}) = \{\text{id}, \sigma\}$.
- Let $f(x) = x^2 + px + q \in \mathbb{Q}[x]$ irreducible with distinct roots θ, θ' . Then $\text{Aut}(\mathbb{Q}(\theta)/\mathbb{Q}) = \{\text{id}, \sigma\} \cong \mathbb{Z}/2$ where $\sigma(a + b\theta) = a + b\theta'$.
- Let θ root of $x^3 - 2$, let $\sigma \in \text{Aut}(\mathbb{Q}(\theta)/\mathbb{Q})$. Now $\sigma(\theta)^3 = \sigma(\theta^3) = \sigma(2) = 2$ so $\sigma(\theta) \in \{\theta, \omega\theta, \omega^2\theta\}$ but $\omega\theta, \omega^2\theta \notin \mathbb{Q}(\theta)$ so $\sigma(\theta) = \theta \implies \sigma = \text{id}$.
- Let $\theta^p = t$, $\sigma \in \text{Aut}(\mathbb{F}_p(\theta)/\mathbb{F}_p(t))$. Then

$$\sigma(\theta)^p = \sigma(\theta^p) = \sigma(t) = t = \theta^p$$

so $(\sigma(\theta) - \theta)^p = \sigma(\theta)^p - \theta^p = 0 \implies \sigma(\theta) = \theta \implies \sigma = \text{id}$.

- Let $\sigma \in \text{Aut}(\mathbb{R}/\mathbb{Q})$. Then $\alpha \leq \beta \in \mathbb{R} \implies \beta - \alpha = \gamma^2, \gamma \in \mathbb{R}$, so $\sigma(\beta) - \sigma(\alpha) = \sigma(\gamma)^2 \geq 0$ so $\sigma(\alpha) \leq \sigma(\beta)$. Given $\alpha \in \mathbb{R}$, there exist sequences $(r_n), (s_n) \subset \mathbb{Q}$ with $r_n \leq \alpha \leq s_n$ and $r_n \rightarrow \alpha, s_n \rightarrow \alpha$ as $n \rightarrow \infty$. Hence $r_n = \sigma(r_n) \leq \sigma(\alpha) \leq \sigma(s_n) = s_n$ so $\sigma(\alpha) = \alpha$ by squeezing. Hence $\text{Aut}(\mathbb{R}/\mathbb{Q}) = \{\text{id}\}$.

Theorem. Let $L = K(\theta)$, θ root of irreducible $f(x) \in K[x]$, $\deg(f) = n$. Then $|\text{Aut}(L/K)| \leq n$, with equality iff $f(x)$ has n distinct roots in L .

Theorem. Let L/K be finite extension. Then $|\text{Aut}(L/K)| \leq [L : K]$, with equality iff L/K is normal and minimal polynomial of every $\theta \in L$ over K has no repeated roots (in a splitting field).

4.3. Separable extensions

Definition. Let L/K finite extension.

- $\theta \in L$ is **separable over K** if its minimal polynomial over K has no repeated roots (in its splitting field).
- L/K is **separable** if every $\theta \in L$ is separable over K .

Example. Let $K = \mathbb{F}_p(t)$, then $f(x) = x^p - t \in K[x]$ is irreducible by Eisenstein's criterion with $p = t$, and $f(x) = x^p - \theta^p = (x - \theta)^p$ so θ is root of multiplicity $p \geq 2$. So $\mathbb{F}_p(\theta)/\mathbb{F}_p(t)$ is normal but not separable.

Definition. Let $f(x) = \sum_{i=0}^n a_i x^i \in K[x]$. **Formal derivative** of $f(x)$ is

$$Df(x) = D(f) := \sum_{i=1}^n i a_i x^{i-1} \in K[x]$$

Note. Formal derivative satisfies:

$$D(f + g) = D(f) + D(g), \quad D(fg) = f \cdot D(g) + D(f) \cdot g, \quad \forall a \in K, D(a) = 0$$

Also $\deg(D(f)) < \deg(f)$. But if $\text{char}(K) = p$, then $D(x^p) = px^{p-1} = 0$ so it is not always true that $\deg(D(f)) = \deg(f) - 1$.

Note. If $f(x)$ has a repeated root α , then $Df(\alpha) = 0$.

Theorem (sufficient conditions for separability). Finite extension L/K is separable if any of the following hold:

- $\text{char}(K) = 0$,
- $\text{char}(K) = p$ and $K = \{b^p : b \in K\} = K^p$ for prime p ,
- $\text{char}(K) = p$ and $p \nmid [L : K]$

Definition. K is **perfect field** if either of first two of above properties hold.

Remark. All finite extensions of any perfect extension (e.g. \mathbb{Q}, \mathbb{F}_p) are separable (recall Fermat's little theorem: $\forall a \in \mathbb{F}_p, a = a^p$). So to find a non-separable extension L/K , we need $\text{char}(K) = p > 0$, K infinite and $p \mid [L : K]$. For example, $L = \mathbb{F}_p(\theta)$, $K = \mathbb{F}_p(t)$ where $\theta^p = t$.

Theorem. Let $\alpha_1, \dots, \alpha_n$ algebraic over K , then $K(\alpha_1, \dots, \alpha_n)/K$ is separable iff every α_i is separable over K .

Remark. For tower $K \subseteq M \subseteq L$, L/K is separable iff L/M and M/K are separable. However, the same statement for normality does not hold.

Theorem (Theorem of the Primitive Element). Let L/K finite and separable. Then L/K is simple, i.e. $\exists \alpha \in L : L = K(\alpha)$.

4.4. The fundamental theorem of Galois theory

Definition. Finite extension L/K is **Galois extension** if it is normal and separable. Equivalently, $|\text{Aut}(L/K)| = [L : K]$. When L/K is Galois, the **Galois group** is $\text{Gal}(L/K) := \text{Aut}(L/K)$.

Definition. Let $\mathcal{F} := \{\text{intermediate fields of } L/K\}$ and $\mathcal{G} := \{\text{subgroups of } \text{Gal}(L/K)\}$. Define the map $\Gamma : \mathcal{F} \rightarrow \mathcal{G}$, $\Gamma(M) = \text{Gal}(L/M)$.

Definition. Let L field, G a group of automorphisms of L . **Fixed field** L^G of G is set of elements in L which are invariant under all automorphisms in G :

$$L^G := \{\alpha \in L : \forall \sigma \in G, \sigma(\alpha) = \alpha\}$$

Theorem. If G is finite group of automorphisms of L then L^G is subfield of L and $[L : L^G] = |G|$.

Corollary. If L/K is Galois then

- $L^{\text{Gal}(L/K)} = K$.
- If $L^G = K$ for some group G of K -automorphisms of L , then $G = \text{Gal}(L/K)$.

Note. Let $\sigma \in \text{Gal}(L/K)$. If $\alpha \in L$ has minimal polynomial $f(x) \in K[x]$ over K , then $f(\sigma(\alpha)) = 0$, and

$$f(\sigma(\alpha)) = \sigma(f(\alpha))$$

by properties of field homomorphisms. Hence $\sigma(\alpha)$ is also a root of $f(x)$ for any $\sigma \in \text{Gal}(L/K)$, i.e. σ permutes the roots of $f(x)$.

Remark. If L/K is Galois and $\alpha \in L$ but $\alpha \notin K$, then there exists an automorphism $\sigma \in \text{Gal}(L/K)$ such that $\sigma(\alpha) \neq \alpha$.

Definition. For H subgroup of $\text{Gal}(L/K)$, set $L^H := \{\alpha \in L : \forall \sigma \in H, \sigma(\alpha) = \alpha\}$, then $K \subseteq L^H \subseteq L$. Define $\Phi : \mathcal{G} \rightarrow \mathcal{F}$, $\Phi(H) = L^H$.

- Γ and Φ are inclusion-reversing: $M_1 \subseteq M_2 \implies \Gamma(M_2) \subseteq \Gamma(M_1)$, and $H_1 \subseteq H_2 \implies \Phi(H_2) \subseteq \Phi(H_1)$.

Theorem (Fundamental theorem of Galois theory - Theorem A). For finite Galois extension L/K ,

- $\Gamma : \mathcal{F} \rightarrow \mathcal{G}$ and $\Phi : \mathcal{G} \rightarrow \mathcal{F}$ are mutually inverse bijections (the **Galois correspondence**).
- For $M \in \mathcal{F}$, L/M is Galois and $|\text{Gal}(L/M)| = [L : M]$.
- For $H \in \mathcal{G}$, L/L^H is Galois and $\text{Gal}(L/L^H) = H$.

Remark. $\text{Gal}(L/K)$ acts on \mathcal{F} : given $\sigma \in \text{Gal}(L/K)$ and $K \subseteq M \subseteq L$, consider $\sigma(M) = \{\sigma(\alpha) : \alpha \in M\}$ which is a subfield of L and contains K , since σ fixes elements of K . Given another automorphism $\tau : L \rightarrow L$,

$$\begin{aligned}
\tau \in \text{Gal}(L/\sigma(M)) &\iff \forall \alpha \in M, \tau(\sigma(\alpha)) = \sigma(\alpha) \\
&\iff \forall \alpha \in M, \sigma^{-1}(\tau(\sigma(\alpha))) = \alpha \\
&\iff \sigma^{-1}\tau\sigma \in \text{Gal}(L/M) \\
&\iff \tau \in \sigma \text{Gal}(L/M)\sigma^{-1}
\end{aligned}$$

Hence $\sigma \text{Gal}(L/M)\sigma^{-1}$ and $\text{Gal}(L/M)$ are conjugate subgroups of $\text{Gal}(L/K)$. Now

$$[M : K] = \frac{[L : K]}{[L : M]} = \frac{|\text{Gal}(L/K)|}{|\text{Gal}(L/M)|}$$

Theorem (Fundamental theorem of Galois theory - Theorem B). Let L/K be finite Galois extension, $G = \text{Gal}(L/K)$ and $K \subseteq M \subseteq L$. Then the following are equivalent:

- M/K is Galois.
- $\forall \sigma \in G, \sigma(M) = M$.
- $H = \text{Gal}(L/M)$ is normal subgroup of $G = \text{Gal}(L/K)$.

When these conditions hold, we have $\text{Gal}(M/K) \cong G/H$.

Example. Let L/K be Galois, $[L : K] = p$ prime.

- By the tower law, any $K \subseteq M \subseteq L$ has $[L : M] \in \{1, p\}$, $[M : K] \in \{p, 1\}$, so $M = L$ or K . In both cases, M/K is normal.
- $|\text{Gal}(L/K)| = [L : K] = p$ so $\text{Gal}(L/M) \cong \mathbb{Z}/p$, so the only subgroups are $\text{Gal}(L/K)$ and $\{\text{id}\}$. In both cases, H is normal subgroup of $\text{Gal}(L/K)$.

4.5. Computations with Galois groups

Example (quadratic extension). $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is normal (since degree is 2) and separable (since characteristic is zero). Any element of $\varphi \in G = \text{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ is determined by the image of $\sqrt{2}$. But $\varphi(\sqrt{2})^2 = \varphi(2) = 2$ so $\varphi(\sqrt{2}) = \pm\sqrt{2}$. This gives two automorphisms $\text{id}(\sqrt{2}) = \sqrt{2}$ and $\sigma(\sqrt{2}) = -\sqrt{2}$. So $G = \{\text{id}, \sigma\} = \langle \sigma \rangle \cong \mathbb{Z}/2$. Subgroup $\{\text{id}\}$ corresponds to $\mathbb{Q}(\sqrt{2})$, G corresponds to \mathbb{Q} .

Example (biquadratic extension). $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ over \mathbb{Q} is normal (as splitting field of $(x^2 - 2)(x^2 - 3)$ over \mathbb{Q}) and separable (as $\text{char}(\mathbb{Q}) = 0$), so is Galois extension. Let σ be given as before.

- Suppose $\sqrt{3} \in \mathbb{Q}(\sqrt{2})$, then $\sigma(\sqrt{3})^2 = \sigma(3) = 3$, so $\sigma(\sqrt{3}) = \pm\sqrt{3}$.
- If $\sigma(\sqrt{3}) = \sqrt{3}$, then $\sqrt{3} \in \mathbb{Q}(\sqrt{2})^{\{\text{id}, \sigma\}} = \mathbb{Q}$: contradiction.
- If $\sigma(\sqrt{3}) = -\sqrt{3}$, then $\sigma(\sqrt{2})\sigma(\sqrt{3}) = \sigma(\sqrt{6}) = (-\sqrt{2})(-\sqrt{3}) = \sqrt{6}$, so $\sqrt{6} \in \mathbb{Q}(\sqrt{2})^{\{\text{id}, \sigma\}} = \mathbb{Q}$: contradiction.
- So $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$, hence $[L : \mathbb{Q}] = [L : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 4$.
- Now $G = \text{Gal}(L/\mathbb{Q})$ has order $[L : \mathbb{Q}] = 4$, so $G \cong \mathbb{Z}/4$ or $\mathbb{Z}/2 \times \mathbb{Z}/2$.
- For $\varphi \in G$, $\varphi(\sqrt{2})^2 = 2 \implies \varphi(\sqrt{2}) = \pm\sqrt{2}$, $\varphi(\sqrt{3})^2 = 3 \implies \varphi(\sqrt{3}) = \pm\sqrt{3}$. So there are four choices, corresponding to choices of \pm signs.
- Define σ, τ by $\sigma(\sqrt{2}) = -\sqrt{2}$, $\sigma(\sqrt{3}) = \sqrt{3}$, $\tau(\sqrt{2}) = \sqrt{2}$, $\tau(\sqrt{3}) = -\sqrt{3}$. Now $\sigma^2 = \tau^2 = \text{id}$, $\sigma\tau(\sqrt{2}) = -\sqrt{2}$, $\sigma\tau(\sqrt{3}) = -\sqrt{3}$ and $\sigma\tau = \tau\sigma$.
- So $G = \langle \sigma, \tau : \sigma^2 = \tau^2 = \text{id}, \sigma\tau = \tau\sigma \rangle = \langle \sigma \rangle \times \langle \tau \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$.
- G has proper subgroups $H_1 = \langle \sigma \rangle$, $H_2 = \langle \tau \rangle$, $H_3 = \langle \sigma\tau \rangle$.
- So the intermediate fields are $L^{H_1}, L^{H_2}, L^{H_3}$.

- $\sigma(\sqrt{3}) = \sqrt{3} \implies \sqrt{3} \in L^{H_1}$ so $\mathbb{Q}(\sqrt{3}) \subseteq L^{H_1}$, but $[L : \mathbb{Q}(\sqrt{3})] = 2 = |H_1| = [L : L^{H_1}]$. Hence $L^{H_1} = \mathbb{Q}(\sqrt{3})$. Similarly $L^{H_2} = \mathbb{Q}(\sqrt{2})$.
- $\sigma\tau(\sqrt{6}) = \sqrt{6} \implies \sqrt{6} \in L^{H_3}$, so $L^{H_3} = \mathbb{Q}(\sqrt{6})$.

Remark. It is not generally true that $[K(\sqrt{a}, \sqrt{b}) : K] = 4$, e.g. $\mathbb{Q}(\sqrt{2}, \sqrt{8}) = \mathbb{Q}(\sqrt{2})$.

Remark. Can generalise above example to arbitrary $K(\sqrt{a}, \sqrt{b})/K$ where $\text{char}(K) \neq 2$, and $a, b \in K$, $a, b, ab \notin (K^\times)^2$ where $(K^\times)^2$ is set of squares of K^\times .

Example (degree 8 extension).

- Consider $L = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ over \mathbb{Q} . L is splitting field of $(x^2 - 2)(x^2 - 3)(x^2 - 5)$, so is normal, and $\text{char}(\mathbb{Q}) = 0$, so is separable, so is Galois.
- Let $M = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. By above, $\text{Gal}(M/\mathbb{Q}) = \langle \sigma \rangle \times \langle \tau \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$.
- Suppose $\sqrt{5} \in M$. Then $\sigma(\sqrt{5})^2 = \tau(\sqrt{5})^2 = 5$, so $\sigma(\sqrt{5}) = \pm\sqrt{5}$, $\tau(\sqrt{5}) = \pm\sqrt{5}$.
- If $\sigma(\sqrt{5}) = \sqrt{5}$, then $\sqrt{5} \in M^{(\sigma)} = \mathbb{Q}(\sqrt{3})$.
 - If $\tau(\sqrt{5}) = \sqrt{5}$, $\sqrt{5} \in M^{(\sigma, \tau)} = \mathbb{Q}$: contradiction.
 - If $\tau(\sqrt{5}) = -\sqrt{5}$, then since $\sqrt{15} \in M^{(\sigma)}$, $\tau(\sqrt{15}) = \sqrt{15}$, so $\sqrt{15} \in M^{(\sigma, \tau)} = \mathbb{Q}$: contradiction.
- If $\sigma(\sqrt{5}) = -\sqrt{5}$, then $\sigma(\sqrt{10}) = \sigma(\sqrt{2})\sigma(\sqrt{5}) = (-\sqrt{2})(-\sqrt{5}) = \sqrt{10}$, so $\sqrt{10} \in M^{(\sigma)} = \mathbb{Q}(\sqrt{3})$.
 - If $\tau(\sqrt{5}) = \sqrt{5}$, $\tau(\sqrt{10}) = \sqrt{10} \in M^{(\sigma, \tau)} = \mathbb{Q}$: contradiction.
 - If $\tau(\sqrt{5}) = -\sqrt{5}$, $\tau(\sqrt{30}) = \tau(\sqrt{5})\tau(\sqrt{3})\tau(\sqrt{2}) = \sqrt{30} \in M^{(\sigma, \tau)} = \mathbb{Q}$: contradiction.
- So $\sqrt{5} \notin M$, so $[L : \mathbb{Q}] = [L : M][M : \mathbb{Q}] = 8$. The 8 elements in $\text{Gal}(L/\mathbb{Q})$ are determined by choices of $\sqrt{a} \mapsto \pm\sqrt{a}$ where $a \in \{2, 3, 5\}$.
- $\text{Gal}(L/\mathbb{Q}) = \langle \sigma_1, \sigma_2, \sigma_3 \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ where $\sigma_1(\sqrt{2}) = -\sqrt{2}$, $\sigma_2(\sqrt{3}) = -\sqrt{3}$, $\sigma_3(\sqrt{5}) = -\sqrt{5}$ and the σ_i fix all other square roots.
- More generally, write $\sigma(\sqrt{5}) = (-1)^j \sqrt{5}$, $\tau(\sqrt{5}) = (-1)^k \sqrt{5}$, $j, k \in \{0, 1\}$. Define $m = 2^j 3^k$, then $\sigma(\sqrt{m}) = (-1)^j \sqrt{m} \implies \sigma(\sqrt{5m}) = \sqrt{5m}$ and $\tau(\sqrt{m}) = (-1)^k \sqrt{m} \implies \tau(\sqrt{5m}) = \sqrt{5m}$, so $\sqrt{5m} \in M^{(\sigma, \tau)} = \mathbb{Q}$: contradiction.

Example (cubic extension and its normal closure).

- Let $L = \mathbb{Q}(\theta)$, $\theta^3 - 2 = 0$. L/\mathbb{Q} isn't Galois since not normal. Take the normal closure $N = \mathbb{Q}(\theta, \omega) = \mathbb{Q}(\theta, \sqrt{-3})$.
- Let $M = \mathbb{Q}(\omega)$ so $[M : \mathbb{Q}] = 2$, $[L : \mathbb{Q}] = 3$ and $[N : \mathbb{Q}] = 6$. Let $G = \text{Gal}(N/\mathbb{Q})$.
- Since $|G| = [N : \mathbb{Q}] = 6$, $G \cong \mathbb{Z}/6$ or $G \cong D_3 \cong S_3$.
- G contains $\text{Gal}(N/L)$. Since $N = L(\omega)$,

$$\text{Gal}(N/L) = \{\text{id}, \tau\} = \langle \tau \rangle \cong \mathbb{Z}/2$$

where $\tau(\sqrt{-3}) = -\sqrt{-3}$ (i.e. $\tau(\omega) = \omega^2$) and $\tau(\theta) = \theta$ as $\theta \in L$.

- G contains $H = \text{Gal}(N/M)$. $N = M(\theta)$, $|H| = [N : M] = 3$ so $\text{Gal}(N/M)$ is cyclic so

$$H = \{\text{id}, \sigma, \sigma^2\} = \langle \sigma \rangle \cong \mathbb{Z}/3$$

where $\sigma(\theta) = \omega\theta$, also $\sigma(\omega) = \omega$ as $\omega \in M$ and $\sigma^2(\theta) = \omega^2\theta$, so H permutes the three roots of $x^3 - 2$.

- $\tau \notin H$ so $H = \{\text{id}, \sigma, \sigma^2\}$ and $\tau H = \{\tau, \tau\sigma, \tau\sigma^2\}$ are disjoint cosets. So $G = H \cup \tau H = \langle \tau, \sigma \rangle$ so $|G| = 6$. $\tau^2 = \sigma^3 = \text{id}$ and $\sigma\tau = \tau\sigma^2$. So $G \cong S_3 \cong D_3$.
- G has one subgroup of order 3, $H = \langle \sigma \rangle$. Fixed field is $N^H = M$. H is only proper normal subgroup of G . Correspondingly, M is only normal extension of Q in N .
- There are 3 order 2 subgroups: $\langle \tau \rangle, \langle \tau\sigma \rangle, \langle \tau\sigma^2 \rangle$. $N^{\langle \tau \rangle} = \mathbb{Q}(\theta) = L$, $N^{\langle \tau\sigma \rangle} = \mathbb{Q}(\omega\theta) = \sigma(L)$, $N^{\langle \tau\sigma^2 \rangle} = \mathbb{Q}(\omega^2\theta) = \sigma^2(L)$.

Example. Show $\sqrt[3]{3} \notin \mathbb{Q}(\sqrt[3]{2})$.

- Assume $\sqrt[3]{3} \in \mathbb{Q}(\sqrt[3]{2})$. Then $\sqrt[3]{3} \in N = \mathbb{Q}(\omega, \sqrt[3]{2})$, the normal closure.
- As above, let $\sigma \in \text{Gal}(N/\mathbb{Q})$, $\sigma(\sqrt[3]{2}) = \omega\sqrt[3]{2}$ and $N^{\langle \sigma \rangle} = \mathbb{Q}(\omega)$. Also,

$$\sigma(\sqrt[3]{3})^3 = \sigma(3) = 3 \implies \sigma(\sqrt[3]{3}) \in \{\sqrt[3]{3}, \omega\sqrt[3]{3}, \omega^2\sqrt[3]{3}\}$$

- If $\sigma(\sqrt[3]{3}) = \sqrt[3]{3}$, then $\sqrt[3]{3} \in N^{\langle \sigma \rangle} = \mathbb{Q}(\omega)$, so $\mathbb{Q}(\sqrt[3]{3}) \subseteq \mathbb{Q}(\omega)$: contradiction.
- If $\sigma(\sqrt[3]{3}) = \omega\sqrt[3]{3}$, then $\sigma(\sqrt[3]{3}/\sqrt[3]{2}) = \sqrt[3]{3}/\sqrt[3]{2}$ hence $\sqrt[3]{3}/2 \in N^{\langle \sigma \rangle} = \mathbb{Q}(\omega)$, so $\mathbb{Q}(\sqrt[3]{3}/2) = \mathbb{Q}(\sqrt[3]{12}) \subseteq \mathbb{Q}(\omega)$: contradiction.
- If $\sigma(\sqrt[3]{3}) = \omega^2\sqrt[3]{3}$, $\mathbb{Q}(\sqrt[3]{3}/4) = \mathbb{Q}(\sqrt[3]{6}) \subseteq \mathbb{Q}(\omega)$: contradiction.

Remark. In the above example, $N = \mathbb{Q}(\theta_1, \theta_2, \theta_3) = \mathbb{Q}(\sqrt[3]{2}, \omega)$ where θ_i are the roots of $x^3 - 2$. Plotting these roots on Argand diagram gives the symmetry group $S_3 \cong D_3$ of an equilateral triangle. τ reflects the θ_i (complex conjugation), σ rotates the roots (but **doesn't** rotate all of N , as it fixes \mathbb{Q}). For $g \in G$, $g(\theta_j) = \theta_{\pi(j)}$ where π is permutation of $\{1, 2, 3\}$. So there is a group homomorphism $\varphi : G \rightarrow S_3$, $\varphi(g) = \pi$. $\ker(\varphi) = \{\text{id}\}$, so φ is injective and also surjective, since $|G| = |S_3| = 6$, so φ is isomorphism.

Definition. For $f(x) \in K[x]$, $\deg(f) = n \geq 1$, with n distinct roots, the **Galois group** of $f(x)$, G_f , is Galois group of splitting field of $f(x)$ over K (provided it is separable).

Remark. Elements of G_f permute roots of f , so G_f is subgroup of S_n . If $f(x)$ irreducible over K , then G_f is **transitive** subgroup, i.e. given 2 roots α, β of f , there is a $g \in G_f$ with $g(\alpha) = \beta$. This gives a general pattern

polynomial \longrightarrow field extension \longrightarrow permutation group

Example. Consider $\mathbb{Q} \subset L = \mathbb{Q}(\theta) \subset N = \mathbb{Q}(\theta, i)$ where $\theta = \sqrt[4]{2}$. N is normal closure of $\mathbb{Q}(\theta)$, $[N : \mathbb{Q}] = 8$ so $|\text{Gal}(N/\mathbb{Q})| = 8$.

- Define $\sigma(\theta) = i\theta$, $\sigma(i) = i$, $\tau(\theta) = \theta$, $\tau(i) = -i$. Then $\tau^2 = \sigma^4 = \text{id}$. We have

	id	σ	σ^2	σ^3	τ	$\tau\sigma$	$\tau\sigma^2$	$\tau\sigma^3$
θ	θ	$i\theta$	$-\theta$	$-i\theta$	θ	$-i\theta$	$-\theta$	$i\theta$
i	i	i	i	i	$-i$	$-i$	$-i$	$-i$

so $G = \text{Gal}(N/\mathbb{Q}) = \langle \sigma, \tau : \sigma^4 = \tau^2 = \text{id}, \sigma\tau = \tau\sigma^3 \rangle \cong D_4$.

- Order 2 subgroups are $\langle \tau \rangle, \langle \tau\sigma \rangle, \langle \tau\sigma^2 \rangle, \langle \tau\sigma^3 \rangle, \langle \sigma^2 \rangle$.
- Order 4 subgroups are $\langle \sigma^2, \tau \rangle \cong (\mathbb{Z}/2)^2, \langle \sigma \rangle \cong \mathbb{Z}/4, \langle \sigma^2, \tau\sigma \rangle \cong (\mathbb{Z}/2)^2$.
- Respectively, intermediate field extensions of degree 4 are $\mathbb{Q}(\sqrt[4]{2}), \mathbb{Q}(i\sqrt[4]{2}), \mathbb{Q}(\sqrt{2}, i), \mathbb{Q}((1-i)\sqrt[4]{2}), \mathbb{Q}((1+i)\sqrt[4]{2})$.
- Respectively, intermediate field extensions of degree 2 are $\mathbb{Q}(\sqrt{2}), \mathbb{Q}(i), \mathbb{Q}(i\sqrt{2})$.

5. Cyclotomic field extensions

5.1. Roots of unity

Definition. $\zeta \in K^*$ is n -th primitive root of unity if $\zeta^n = 1$ and $\forall 0 < m < n, \zeta^m \neq 1$, i.e. order of ζ in K^* is n .

Example.

- ζ is primitive 1-st root of unity iff $\zeta = 1$.
- -1 is primitive 2-nd root of unity iff $\text{char}(K) \neq 2$.
- If $\text{char}(K) = p$ prime, then K contains no p -th primitive roots of unity (since $\zeta^p = 1 \iff (\zeta - 1)^p = 0 \iff \zeta = 1$).
- If $K = \mathbb{C}$, $\exp(2\pi i/n)$ is n -th primitive root of unity.

Proposition. Let $\zeta \in K^*$ primitive n -th root of unity, let $d = \text{gcd}(m, n)$. Then ζ^m is primitive (n/d) -th root of unity.

Corollary. Let $\zeta \in K^*$ primitive n -th root of unity.

- $\zeta^m = 1 \iff m \equiv 0 \pmod{n}$.
- ζ^m is primitive n -th root of unity iff $\text{gcd}(m, n) = 1$.

Definition. Let $\mu(K)$ denote subgroup of all roots of unity in K^* .

Theorem. Let K field, H finite subgroup of K^* , then H is cyclic.

Remark. This implies that any finite field \mathbb{F}_q can be written $\mathbb{F}_q = \mathbb{F}_{p^n} = \mathbb{F}_p(\alpha)$ where α is generator of \mathbb{F}_q^\times .

Corollary. Let K field, $n \in \mathbb{N}$ be largest such that K contains primitive n -th root of unity ζ . Then $\mu(K)$ is cyclic subgroup in K^* generated by ζ .

5.2. n -th cyclotomic field extensions

Notation. Let $\zeta_n = \exp(2\pi i/n) \in \mathbb{C}$.

Definition. $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is n -th cyclotomic field extension.

Proposition. $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is Galois.

Definition. $\Phi_n(x) := \prod_{a \in A} (x - \zeta_n^a)$ where $A = \{a \in \mathbb{N} : 0 < a < n, \text{gcd}(a, n) = 1\}$.

Proposition. $\Phi_n(x) \in \mathbb{Q}[x]$ is irreducible and so is minimal polynomial of a primitive n -th root of unity over \mathbb{Q} . In particular, $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(n)$, where $\varphi(n) = |(\mathbb{Z}/n)^\times|$ is Euler function.

Proposition. Properties of φ function:

- For prime p , $\varphi(p) = p - 1$.
- For prime p , $\varphi(p^k) = p^k - p^{k-1}$.
- If $\text{gcd}(n, m) = 1$, then $\varphi(nm) = \varphi(n)\varphi(m)$.

- If $n = \prod_{i=1}^r p_i^{k_i}$ is prime factorisation of n , then

$$\varphi(n) = n \prod_{i=1}^r \left(1 - \frac{1}{p_i}\right)$$

Proposition. $\forall n \in \mathbb{N}, x^n - 1 = \prod_{n_1|n} \Phi_{n_1}(x)$.

Example.

- $\Phi_1(x) = x - 1$.
- $\Phi_1(x)\Phi_2(x) = x^2 - 1 \implies \Phi_2(x) = x + 1$.
- $\Phi_1(x)\Phi_3(x) = x^3 - 1 \implies \Phi_3(x) = x^2 + x + 1$.

Proposition.

- For p prime, $\Phi_p(x) = x^{p-1} + \dots + x + 1$.
- For p prime, $\Phi_{p^k}(x) = \Phi_p(x^{p^{k-1}})$.
- For every $n \in \mathbb{N}$, $\Phi_n(x)$ has integer coefficients.

5.3. Galois properties of cyclotomic extensions

Theorem. $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n)^\times$.

Remark. To compute $(\mathbb{Z}/n)^\times$, note that for m, n coprime, $(\mathbb{Z}/mn)^\times \cong (\mathbb{Z}/m)^\times \times (\mathbb{Z}/n)^\times$ and

- If $p \neq 2$ prime, then $(\mathbb{Z}/p^r)^\times$ is cyclic of order $\varphi(p^r)$.
- $(\mathbb{Z}/4)^\times \cong \mathbb{Z}/2$ and for $r \geq 3$, $(\mathbb{Z}/2^r)^\times \cong \mathbb{Z}/2 \times \mathbb{Z}/2^{r-2}$.

Corollary. $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ is abelian so every subgroup is normal, so any subfield of $\mathbb{Q}(\zeta_n)$ is Galois over \mathbb{Q} .

Corollary. For p prime, $G = \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \cong (\mathbb{Z}/p)^\times \cong \mathbb{Z}/(p-1)$. In particular, for $d \mid (p-1)$, $\mathbb{Q}(\zeta_p)$ contains exactly one subfield of degree d and there are no other subfields.

Remark. For $d = 2$ in above corollary, $\mathbb{Q}(\zeta_p)$ contains unique quadratic subfield $\mathbb{Q}(\sqrt{D_p})$, where $D_p = (-1)^{(p-1)/2} p$

Example. $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ not always cyclic, e.g. $\text{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$.

Proposition.

- If n odd, $\mu(\mathbb{Q}(\zeta_n))$ is cyclic of order $2n$ and is generated by $-\zeta_n$.
- If n even, $\mu(\mathbb{Q}(\zeta_n))$ is cyclic of order n and is generated by ζ_n .
- If $\text{gcd}(m, n) = d$, then $\mathbb{Q}(\zeta_m, \zeta_n) = \mathbb{Q}(\zeta_{mn/d})$.

5.4. Special properties of $\mathbb{Q}(\zeta_p)$, where $p > 2$ is prime

Example. $\text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q}) \cong (\mathbb{Z}/5)^\times$ has generator $\tau : \zeta_5 \mapsto \zeta_5^2$. \mathbb{Q} -basis $\{1, \zeta_5, \zeta_5^2, \zeta_5^3\}$ is not invariant under action of τ or any power of τ (since $\tau(\zeta_5^2) = \zeta_5^4$) but $\{\zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4\}$ is invariant. The same holds for general $p > 2$ prime. For $\alpha_i \in \mathbb{Q}$, $\alpha_1 \zeta_p + \dots + \alpha_{p-1} \zeta_p^{p-1} \in \mathbb{Q}$ iff $\alpha_1 = \dots = \alpha_{p-1}$.

Example. If $x \in \mathbb{Q}(\zeta_p)$, $[\mathbb{Q}(x) : \mathbb{Q}] = |\{\sigma(x) : \sigma \in \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})\}|$ In particular, if τ is generator of $G = \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ and $x = \alpha_1 \zeta_p + \dots + \alpha_{p-1} \zeta_p^{p-1}$ then set of all conjugates of x is equal to (note not all elements are distinct)

$$\{\tau^a(x) : a \in [p-1]\} = \left\{ \sum_{i=1}^{p-1} \alpha_i \zeta_p^{ai} : a \in [p-1] \right\}$$

Example. Let $x = \zeta_5 + \zeta_5^4$, $\tau : \zeta_5 \mapsto \zeta_5^2$ is a generator of $\text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$. $\tau(x) = \zeta_5^2 + \zeta_5^3 \neq x$ but $\tau^2(x) = x$, so $[\mathbb{Q}(x) : \mathbb{Q}] = 2$, i.e. $\mathbb{Q}(\zeta_5 + \zeta_5^4)$ is unique quadratic subfield in $\mathbb{Q}(\zeta_5)$.

Definition. Let $x \in \mathbb{Q}(\zeta_p)$, let minimal polynomial of x over \mathbb{Q} be $m(t) = (t - x^{(1)}) \cdots (t - x^{(d)})$. **Conjugates** of x over \mathbb{Q} are $x^{(1)} = x, \dots, x^{(d)}$.

Example. Minimal polynomial of $\zeta_5 + \zeta_5^4 = 2 \cos(2\pi/5)$ over \mathbb{Q} is $m(x) = (x - \zeta_5 - \zeta_5^4)(x - \zeta_5^2 - \zeta_5^3) = x^2 + x - 1$, with roots $(-1 \pm \sqrt{5})/2$. So $\cos(2\pi/5) = (-1 + \sqrt{5})/4$, and unique quadratic subfield of $\mathbb{Q}(\zeta_5)$ over \mathbb{Q} is $\mathbb{Q}(\sqrt{5})$.

Example. Let $\tau \in G$ be generator of $G = \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$, i.e. $\tau(\zeta_p) = \zeta_p^a$, $a \bmod p$ generates $(\mathbb{Z}/p)^\times$. Let

$$\Theta_p = \zeta_p - \tau(\zeta_p) + \tau^2(\zeta_p) - \cdots + \tau^{p-3}(\zeta_p) - \tau^{p-2}(\zeta_p)$$

Θ_p behaves like $\sqrt{D_p}$: $\tau(\Theta_p) = -\Theta_p$, $\tau^2(\Theta_p) = \Theta_p$. So $\Theta_p \in \mathbb{Q}(\zeta_p)^{\langle \tau^2 \rangle}$. Also, $\tau(\Theta_p^2) = \Theta_p^2$ so $\Theta_p^2 \in \mathbb{Q}(\zeta_p)^{\langle \tau \rangle} = \mathbb{Q}$. In fact, $\Theta_p^2 = D_p$. Therefore

$$\Theta_p^2 = A + B(\zeta_p + \cdots + \zeta_p^{p-1}) = A - B$$

So when computing Θ_p^2 , only need to consider coefficients for 1 and ζ_p .

6. Cyclic field extensions

6.1. Cyclic extensions of degree 2

Example. Let L/K cyclic of degree 2, so $\text{Gal}(L/K) = \{e, \tau\}$, $\tau^2 = e$. Let $\theta \in L - K$, then $\tau(\theta) \neq \theta$ (as otherwise $\theta \in L^{\langle \tau \rangle} = K$). Let $\theta_1 = \tau(\theta) - \theta$, so $\tau(\theta_1) = \tau^2(\theta) - \tau(\theta) = -\theta_1$. If $\text{char}(K) \neq 2$, then $\theta_1 \neq -\theta_1$ and so $\theta_1 \notin K$, $L = K(\theta_1)$. θ_1 is “better” than θ , since $\tau(\theta_1) = -\theta_1$. Now if $a = \theta_1^2$, then $\tau(a) = a$, so $L = K(\sqrt{a})$.

Theorem. If $\text{char}(K) \neq 2$ and L/K is cyclic quadratic extension, then

$$\exists a \in K^\times - K^{\times 2} : L = K(\sqrt{a})$$

Definition. a_1, \dots, a_n are **independent modulo $K^{\times 2}$** (independent modulo squares) if

$$a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n} \in K^{\times 2} \iff \text{all } \varepsilon_i \text{ are even}$$

Proposition. If $\text{char}(K) \neq 2$:

- $K(\sqrt{a_1}) = K(\sqrt{a_2}) \iff a_1 \equiv a_2 \pmod{K^{\times 2}}$, i.e. $a_1 = a_2 \cdot b^2$, $b \in K^\times$.
- If $a_1, \dots, a_n \in K^\times$ are independent modulo $K^{\times 2}$ then $K(\sqrt{a_1}, \dots, \sqrt{a_n})$ has degree 2^n over K with Galois group $\cong (\mathbb{Z}/2)^n$.
- If L/K Galois with Galois group $(\mathbb{Z}/2)^n$, then

$$\exists a_1, \dots, a_n \in K^\times : L = K(\sqrt{a_1}, \dots, \sqrt{a_n})$$

Remark. Let $\text{char}(K) = 2$, then $\forall a \in K^\times$, $L = K(\sqrt{a})$ is normal but not separable (since minimal polynomial of e.g. \sqrt{a} is $x^2 - a = (x + \sqrt{a})(x - \sqrt{a}) = (x - \sqrt{a})^2$ so has repeated roots).

6.2. Cyclic extensions of degree n (the Kummer theory)

Definition. L/K is **cyclic of degree n** if it is Galois and $\text{Gal}(L/K)$ is cyclic of order n .

Theorem. If K contains primitive n -th root of unity and for all divisors $d > 1$ of n , $a \in K^\times$ is not d -th power in K , then $L = K(\sqrt[n]{a})$ is cyclic extension of K of degree n . In particular, $x^n - a \in K[x]$ is irreducible.

Proposition. If $\zeta_p \in K$, $a \in K^\times - K^{\times p}$, then $K(\sqrt[p]{a})/K$ is cyclic of degree p . In particular, $x^p - a \in K[x]$ is irreducible.

Theorem. Let K contain primitive n -th root of unity ζ_n , L/K is cyclic extension of degree n , $\text{Gal}(L/K) = \langle \sigma \rangle$. Then

$$\exists a \in K^\times : L = K(\sqrt[n]{a})$$

Such an a is given by θ_b^n for some $b \in L$, where

$$\theta_b = b + \zeta_n^{-1}\sigma(b) + \dots + \zeta_n^{-(n-1)}\sigma^{n-1}(b)$$

is **Lagrange resolvent** for b , i.e. $L = K(\theta_b)$.

Lemma (Artin's lemma). There exists $b \in L$ such that $\theta_b \neq 0$.

7. Finite fields

7.1. Existence and uniqueness

Lemma. Let K finite field, then K is field extension of \mathbb{F}_p for some prime p and $|K| = p^n$ where $n = [K : \mathbb{F}_p]$.

Theorem. Let p prime. Then $\forall n \in \mathbb{N}$, there is field K with $|K| = p^n$.

Theorem. Let K finite field with $|K| = q = p^n$. Then

- $\forall \alpha \in K, \alpha^q = \alpha$.
- $x^q - x = \prod_{\alpha \in K} (x - \alpha)$
- K is splitting field of $x^q - x$ over \mathbb{F}_p .

Corollary. If K_1, K_2 finite fields, $|K_1| = |K_2|$, then $K_1 \cong K_2$.

Definition. Let $q = p^n$, then \mathbb{F}_q is the unique (up to isomorphism) field containing q elements.

Definition. For $q = p^n$, the **Frobenius automorphism** is

$$\sigma : \mathbb{F}_q \rightarrow \mathbb{F}_q, \quad \sigma(\alpha) = \alpha^p$$

which is an \mathbb{F}_p -automorphism by Fermat's little theorem.

Theorem. Let $q = p^n$, p prime.

- $\mathbb{F}_q/\mathbb{F}_p$ is Galois of degree n .

- Frobenius automorphism generates $\text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ and there is group isomorphism

$$\text{Gal}(\mathbb{F}_q/\mathbb{F}_p) \leftrightarrow \mathbb{Z}/n, \quad \sigma \leftrightarrow 1 \pmod n$$

7.2. Counting irreducible polynomials over finite fields

Notation. Let $\text{Irr}_{\mathbb{F}_p}(m)$ denote set of all irreducible polynomials in $\mathbb{F}_p[x]$ of degree m . Let $N_p(m) = |\text{Irr}_{\mathbb{F}_p}(m)|$.

Theorem. Let $q = p^m$, then $mN_p(m) = |\{\alpha \in \mathbb{F}_q : \mathbb{F}_p(\alpha) = \mathbb{F}_q\}|$.

Remark. To use above theorem, note that $\mathbb{F}_p(\alpha) \neq \mathbb{F}_{p^m}$ iff α belongs to proper subfield of \mathbb{F}_{p^m} .

Example.

- If m is prime, then \mathbb{F}_{p^m} has only one proper subfield \mathbb{F}_p , so $mN_p(m) = |\mathbb{F}_{p^m}| - |\mathbb{F}_p| = p^m - p$.
- The proper subfields of \mathbb{F}_{p^4} are \mathbb{F}_p and \mathbb{F}_{p^2} , but $\mathbb{F}_p \subset \mathbb{F}_{p^2}$, so $4N_p(4) = |\mathbb{F}_{p^4}| - |\mathbb{F}_{p^2}|$.
- $\mathbb{F}_p(\alpha) \neq \mathbb{F}_{p^6}$ iff $\alpha \in \mathbb{F}_{p^3} \cup \mathbb{F}_{p^2}$. Since $\mathbb{F}_{p^3} \cap \mathbb{F}_{p^2} = \mathbb{F}_p$, we have $6N_p(6) = |\mathbb{F}_{p^6}| - |\mathbb{F}_{p^3}| - |\mathbb{F}_{p^2}| + |\mathbb{F}_p| = p^6 - p^3 - p^2 + p$.

Proposition. We have

$$p^n = \sum_{m|n} mN_p(m)$$

which we can use recursively to compute any $N_p(m)$.

Example. We construct $L = \mathbb{F}_{3^{16}}$ by finding irreducible polynomial of degree 16 in $\mathbb{F}_3[x]$.

- $\mathbb{F}_9 = \mathbb{F}_3(\theta)$ where $\theta^2 + 1 = 0$, $\mathbb{F}_9 = \{a + b\theta : a, b \in \mathbb{F}_3\}$. $K := \mathbb{F}_9$ contains primitive 8-th root of unity since $\mathbb{F}_9^\times \cong \mathbb{Z}/8$.
- L/K is cyclic extension of degree 8, so by Kummer theory there exists $\alpha \in K$ such that $L = K(\sqrt[8]{\alpha})$. α must be element that is not square or fourth power in \mathbb{F}_9 , so we can look for elements that have order 8.
- $\alpha = \theta$ doesn't work since $\theta^2 = -1 \implies \theta^4 = 1$. $\alpha = 1 + \theta$ works since

$$(1 + \theta)^2 = \theta^2 + \theta + 1 = -\theta, \quad (1 + \theta)^4 = \theta^2 = -1, \quad (1 + \theta)^8 = 1$$

so $\alpha = 1 + \theta$ has order 8 in \mathbb{F}_9 .

- So $L = K(\sqrt[8]{\alpha}) = \mathbb{F}_9(\sqrt[8]{1 + \theta}) = \mathbb{F}_3(\theta, \sqrt[8]{1 + \theta}) = \mathbb{F}_3(\eta)$ where $\eta^8 = 1 + \theta$. Now $[L : \mathbb{F}_3] = 16$ by tower law, so $L = \mathbb{F}_{3^{16}}$ by uniqueness of finite fields.
- $\eta^8 = 1 + \theta \implies (\eta^8 - 1)^2 = \theta^2 = -1 \implies \eta^{16} + \eta^8 + 2 = 0$ so $f(x) = x^{16} + x^8 + 2 \in \mathbb{F}_3[x]$ is irreducible.

8. Galois groups of polynomials

8.1. Symmetric functions

Definition. Define action of S_n on $L = k(x_1, \dots, x_n)$ by $\tau : x_j \mapsto x_{\pi(j)}$ where $\pi \in S_n$, which gives k -automorphism

$$\tau : L \rightarrow L, \quad \frac{f(x_1, \dots, x_n)}{g(x_1, \dots, x_n)} \mapsto \frac{f(x_{\pi(1)}, \dots, x_{\pi(n)})}{g(x_{\pi(1)}, \dots, x_{\pi(n)})}$$

The **symmetric functions** in L are elements of fixed field L^{S_n} .

Definition. The **elementary symmetric polynomials** $e_r \in L$ for $r \in [n]$ are

$$\begin{aligned} e_1 &= \sum_{1 \leq i \leq n} x_i \\ e_2 &= \sum_{1 \leq i < j \leq n} x_i x_j \\ &\vdots \\ e_r &= \sum_{1 \leq i_1 < \dots < i_r \leq n} x_{i_1} \cdots x_{i_r} \\ &\vdots \\ e_n &= x_1 \cdots x_n \end{aligned}$$

Define $K = k(e_1, \dots, e_n)$.

Theorem. $K = L^{S_n}$ and L/K is Galois with $\text{Gal}(L/K) \cong S_n$.

Proof.

- Note that $f(x) = (x - x_1) \cdots (x - x_n) = x^n - e_1 x^{n-1} + \dots + (-1)^n e_n$.
- Show L splitting field of $f(x)$ over K and $[L : K] \leq n!$.
- Show $[L : K] \geq n!$.

□

Remark. Every finite group G is subgroup of S_n for some n , so there is always Galois extension with Galois group G : let $L = k(x_1, \dots, x_n)$, let $G \subseteq S_n$ act on L as above, then $\text{Gal}(L/L^G) = G$.

Definition. For $f(x) \in K[x]$, **Galois group** of $f(x)$, G_f , is Galois group of splitting field of $f(x)$ over K (provided this extension is separable). If $\deg(f) = n$, G_f acts by permuting roots $\theta_1, \dots, \theta_n$ of f , so is subgroup of S_n . There can be non-trivial relationships between roots, so G_f may be proper subgroup.

Corollary. Any symmetric polynomial in $k[x_1, \dots, x_n]$ can be expressed as polynomial in elementary symmetric polynomials, i.e.

$$k[x_1, \dots, x_n]^{S_n} = k[e_1, \dots, e_n]$$

where LHS is set of symmetric polynomials, RHS is set of polynomials in elementary symmetric polynomials.

Example.

- When $n = 2$, $x_1^2 + x_2^2 = e_1^2 - 2e_2$ and $x_1^3 + x_2^3 = e_1^3 - 3e_1e_2$.
- When $n = 3$, $x_1^2x_2 + x_1x_2^2 + x_2^2x_3 + x_2x_3^2 + x_3^2x_1 + x_3x_1^2 = e_1e_2 - 3e_3$.

Definition. **Lexicographic ordering of monomials**, $>_{\text{lex}}$ (or \succ_L), is

$$x_1^{a_1} \cdots x_n^{a_n} >_{\text{lex}} x_1^{b_1} \cdots x_n^{b_n}$$

iff $\exists 0 \leq j \leq n-1$ such that $a_1 = b_1, \dots, a_j = b_j$ and $a_{j+1} > b_{j+1}$.

Example. $x_1^2 x_2^3 x_3 >_{\text{lex}} x_1^2 x_2^2 x_3^4$.

Definition. **Leading term** of $f(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$ is largest monomial $cx_1^{a_1} \dots x_n^{a_n}$ with $c \neq 0$, $a_i \neq 0$ for some i , appearing in f with respect to lexicographic ordering.

Note. If f is symmetric, then $a_1 \geq \dots \geq a_n$.

Algorithm. Given $f(x_1, \dots, x_n) \in k[x_1, \dots, x_n]^{S_n}$, express f as polynomial in elementary symmetric polynomials:

1. Find leading term $cx_1^{a_1} \dots x_n^{a_n}$ of f , compute

$$f_1 = f - ce_1^{a_1 - a_2} \dots e_{n-1}^{a_{n-1} - a_n} e_n^{a_n}$$

Note leading term of $ce_1^{a_1 - a_2} \dots e_{n-1}^{a_{n-1} - a_n} e_n^{a_n}$ is also $cx_1^{a_1} \dots x_n^{a_n}$ so leading term of f_1 is strictly smaller than leading term of f . Also, f_1 is symmetric.

2. If $f_1 \neq 0$, apply step 1 to get f_2, f_3, \dots . Since leading term of f_1, f_2, \dots is strictly decreasing, eventually $f_i = 0$.

Example. Express $f(x_1, x_2) = x_1^3 + x_2^3$ in elementary symmetric polynomials:

- Leading term of f is $x_1^3 = x_1^3 x_2^0$, so

$$f_1 = f - e_1^{3-0} e_2^0 = -3x_1^2 x_2 - 3x_1 x_2^2$$

- Leading term of f_1 is $-3x_1^2 x_2$, so

$$f_2 = f_1 - (-3)e_1^{2-1} e_2^1 = -3x_1^2 x_2 - 3x_1 x_2^2 + 3(x_1 + x_2)x_1 x_2 = 0$$

- So $f_1 = f_2 + (-3)e_1^{2-1} e_2^1 = -3e_1 e_2$ and $f = e_1^3 + f_1 = e_1^3 - 3e_1 e_2$.

Example.

- Let $\theta_1 = \frac{1}{3}(x_1 + \omega x_2 + \omega^2 x_3)$, $\theta_2 = \frac{1}{3}(x_1 + \omega^2 x_2 + \omega x_3)$, where $\omega = \zeta_3$.
- Let $\sigma = (1\ 2\ 3) \in S_3$, then $\sigma(\theta_1) = \omega^2 \theta_1$, $\sigma(\theta_2) = \omega \theta_2$, hence

$$\sigma(\theta_1^3 + \theta_2^3) = \omega^6 \theta_1^3 + \omega^3 \theta_2^3 = \theta_1^3 + \theta_2^3$$

- Let $\tau = (2\ 3) \in S_3$, then $\tau(\theta_1) = \theta_2$, $\tau(\theta_2) = \theta_1$ so $\tau(\theta_1^3 + \theta_2^3) = \theta_1^3 + \theta_2^3$.
- Since $S_3 = \langle \sigma, \tau \rangle$, $f(x_1, x_2, x_3) = 27(\theta_1^3 + \theta_2^3) \in \mathbb{Q}[x_1, x_2, x_3]^{S_3}$. Applying the algorithm:
 - $f_1 = f - 2e_1^3 = 9(x_1^2 x_2 + \dots)$.
 - $f_2 = f_1 - (-9)e_1 e_2 = 27x_1 x_2 x_3$.
 - $f_3 = f_2 - 27e_3 = 0$.
 - So $f = 2e_1^3 - 9e_1 e_2 + 27e_3$.
- By a similar process, $9\theta_1 \theta_2 = e_1^2 - 3e_2$.

8.2. Galois theory for cubic polynomials

Example (Solving quadratic). Let $\text{char}(k) \neq 2$. General quadratic polynomial can be written as

$$f(x) = x^2 - e_1 x + e_2 = (x - x_1)(x - x_2) \in K[x]$$

where $e_1 = x_1 + x_2, e_2 = x_1x_2 \in K = k(e_1, e_2)$. Let $L = k(x_1, x_2) = K(x_1)$, then L/K is Galois and $\text{Gal}(L/K) = \{\text{id}, \sigma\} \cong S_2 \cong \mathbb{Z}/2$ where $\sigma(x_1) = x_2, \sigma(x_2) = x_1$. Since L/K cyclic and $\zeta_2 = -1 \in K$, by [Theorem 6.2.4](#), Lagrange resolvent of x_1 is

$$\theta = \theta_{x_1} = x_1 + \zeta_2^{-1}\sigma(x_1) = x_1 - x_2$$

So $\sigma(\theta) = -\theta$ and $\theta^2 = e_1^2 - 4e_2$. $\Delta = \theta^2$ is **discriminant** of $f(x)$. So we have $x_1 = (e_1 + \sqrt{\Delta})/2, x_2 = (e_1 - \sqrt{\Delta})/2$. If $f(x)$ is irreducible, it has distinct roots, and so Galois group $G_f \cong \mathbb{Z}/2$.

Example (Solving cubic).

- Let $\text{char}(k) \neq 2, 3$, let

$$f(x) = x^3 - e_1x^2 + e_2x - e_3 = (x - x_1)(x - x_2)(x - x_3) \in K[x]$$

where $e_1 = x_1 + x_2 + x_3, e_2 = x_1x_2 + x_1x_3 + x_2x_3, e_3 = x_1x_2x_3 \in K = k(e_1, e_2, e_3) \subset L = K(x_1, x_2, x_3)$.

- By [Theorem 8.1.3](#), $\text{Gal}(L/K) = S_3$ with normal subgroup $A_3 \cong \mathbb{Z}/3$. We have tower $K \subset M = L^{A_3} \subset L$. So $\text{Gal}(L/M) \cong A_3 \cong \mathbb{Z}/3, \text{Gal}(M/K) \cong S_3/A_3 \cong \mathbb{Z}/2$.
- Assume k contains primitive 3rd root of unity ω , so ω^2 is also primitive 3rd root of unity. Define

$$\theta_1 = \frac{1}{3}(x_1 + \omega x_2 + \omega^2 x_3), \quad t_1 = \theta_1^3,$$

$$\theta_2 = \frac{1}{3}(x_1 + \omega^2 x_2 + \omega x_3), \quad t_2 = \theta_2^3$$

then $t_1, t_2 \in M$ and $L = M(\theta_1) = M(\theta_2)$. By [Example 8.1.14](#), $27(\theta_1^3 + \theta_2^3) = 2e_1^3 - 9e_1e_2 + 27e_3, 9\theta_1\theta_2 = e_1^2 - 3e_2$, so t_1, t_2 are roots of **quadratic resolvent** of $f(x)$:

$$(t - t_1)(t - t_2) = t^2 - \left(\frac{2e_1^3 - 9e_1e_2 + 27e_3}{27} \right)t + \left(\frac{e_1^2 - 3e_2}{9} \right)^3$$

- To find roots x_1, x_2, x_3 of f :
 - Solve quadratic resolvent to find t_1, t_2 .
 - Choose $\theta_1 = \sqrt[3]{t_1}$, find θ_2 from $9\theta_1\theta_2 = e_1^2 - 3e_2$.
 - Solve the linear system

$$\begin{cases} x_1 + x_2 + x_3 = e_1 \\ x_1 + \omega x_2 + \omega^2 x_3 = 3\theta_1 \\ x_1 + \omega^2 x_2 + \omega x_3 = 3\theta_2 \end{cases} \implies \begin{cases} x_1 = e_1/3 + \theta_1 + \theta_2 \\ x_2 = e_1/3 + \omega^2\theta_1 + \omega\theta_2 \\ x_3 = e_1/3 + \omega\theta_1 + \omega^2\theta_2 \end{cases}$$

Remark. To solve general cubic $f(x) = x^3 + ax^2 + bx + c$, first perform shift:

$$f(x - a/3) = x^3 + px + q$$

then quadratic resolvent is (*memorise*)

$$t^2 + qt - \frac{p^3}{27}$$

with roots $t_1 = \theta_1^3, t_2 = \theta_2^3$, choose θ_1, θ_2 such that $\theta_1\theta_2 = -\frac{p}{3}$, then roots of $f(x - a/3)$ are $x_1 = \theta_1 + \theta_2, x_2 = \omega^2\theta_1 + \omega\theta_2, \omega\theta_1 + \omega^2\theta_2$.

Example (Galois groups of cubic polynomials). Let $\text{char}(K) \neq 2, 3, f(x) = x^3 + ax^2 + bx + c \in K[x]$, let L be splitting field for $f(x)$ over K , then $G_f = \text{Gal}(L/K)$.

Let $\alpha_1, \alpha_2, \alpha_3$ be roots of $f(x)$ in L .

- If $\alpha_1, \alpha_2, \alpha_3 \in K$, then $L = K, G_f = \{\text{id}\}$.
- If $f(x) = (x - \alpha_j)g(x)$ where $\alpha_j \in K, g(x) \in K[x]$ irreducible quadratic, then $[L : K] = 2, G_f \cong \mathbb{Z}/2$.
- If $f(x)$ irreducible in $K[x]$, then $K \subset K(\alpha_1) \subseteq K(\alpha_1, \alpha_2, \alpha_3) = L$, then either $[L : K(\alpha_1)] = 1$, so $[L : K] = 3$ and $G_f \cong A_3 \cong \mathbb{Z}/3$, or $[L : K(\alpha_1)] = 2$, so $[L : K] = 6$ and $G_f \cong S_3$.

Definition. Discriminant of $f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$ is $\Delta = \delta^2$ where

$$\delta = (\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)$$

Note $\Delta \neq 0$ if f has distinct roots.

Note. If $G_f \cong A_3$, then $G_f = \langle \tau \rangle$ where $\tau : \alpha_1 \mapsto \alpha_2, \alpha_2 \mapsto \alpha_3, \alpha_3 \mapsto \alpha_1$, then $\tau(\delta) = \delta$ so $\delta \in L^{G_f} = K$ and $\Delta \in K^{\times 2}$. But if $G_f \cong S_3$, then if $\tau \in A_3, \tau(\delta) = \delta$ and if $\tau \in S_3 - A_3$, then $\tau(\delta) = -\delta$ so $\delta \notin K$ but $\Delta \in K$.

Theorem. Let $f(x) \in K[x]$ irreducible, $\deg(f) = 3$. Then

- $G_f \cong A_3 \iff \Delta \in K^{\times 2}$,
- $G_f \cong S_3 \iff \Delta \in K^\times - K^{\times 2}$.

Theorem. Let $f(x) = x^3 + ax^2 + bx + c \in K[x]$, then

$$\Delta = 18abc - 4a^3c + a^2b^2 - 4b^3 - 27c^2$$

For reduced cubic $f(x) = x^3 + px + q$, (*memorise*)

$$\Delta = -4p^3 - 27q^2$$

Note. The reduced form of $f(x)$ has same discriminant as $f(x)$.

8.3. Galois theory for quartic polynomials

Example. Let $\text{char}(k) \neq 2, 3, K = k(e_1, e_2, e_3, e_4) \subseteq L = k(x_1, x_2, x_3, x_4)$, so L is splitting field over K of $f(x) = x^4 - e_1x^3 + e_2x^2 - e_3x + e_4$ and $\text{Gal}(L/K) \cong S_4$.

Remark. S_4 can be visualised as symmetries of regular tetrahedron with vertices labelled $\{1, 2, 3, 4\}$. Consider three pairs of opposite edges

$$P_1 = \{(1, 2), (3, 4)\}, \quad P_2 = \{(1, 3), (2, 4)\}, \quad P_3 = \{(1, 4), (2, 3)\}$$

Any permutation in S_4 of the four vertices permutes P_1, P_2, P_3 , which gives map $\pi : S_4 \rightarrow S_3$.

- π is surjective group homomorphism.
- π has kernel $\ker(\pi) = \{\text{id}, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} = V_4 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$.

- $A_4 \subset S_4$ is subgroup of even permutations (orientation-preserving symmetries). Restriction of π to A_4 gives another surjective homomorphism $A_4 \rightarrow A_3$ (and $\pi^{-1}(A_3) = A_4$) also with kernel V_4 .
- V_4 is kernel so is normal subgroup of S_4 and of A_4 . Note that V_4 is only subgroup of A_4 isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$, but there are four subgroups of S_4 , isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$, with V_4 the only normal one.
- This gives increasing sequence of subgroups in S_4

$$\{\text{id}\} \subset \mathbb{Z}/2 \subset V_4 \subset A_4 \subset S_4$$

and $V_4 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, $A_4/V_4 \cong A_3 \cong \mathbb{Z}/3$, $S_4/A_4 \cong \mathbb{Z}/2$.

- Each G_i in this sequence is normal subgroup of G_{i+1} and G_{i+1}/G_i is cyclic, meaning that S_4 is **solvable (soluble) group**.
- We have tower

$$K = L^{S_4} \subset L^{V_4} \subset L = L^{\{e\}}$$

By fundamental theorem, $\text{Gal}(L/L^{V_4}) = V_4 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, so L/L^{V_4} appears as biquadratic extension.

- V_4 is normal in S_4 so by fundamental theorem, $\text{Gal}(L^{V_4}/K) \cong S_4/V_4 \cong S_3$ by first isomorphism theorem. Hence L^{V_4} appears as splitting field of a cubic polynomial over K .

Example (Solving quartic equations). Define

$$\begin{aligned}\theta_1 &= \frac{1}{2}(x_1 + x_2 - x_3 - x_4), \\ \theta_2 &= \frac{1}{2}(x_1 - x_2 + x_3 - x_4), \\ \theta_3 &= \frac{1}{2}(x_1 - x_2 - x_3 + x_4)\end{aligned}$$

Then $\forall j \in [3], \forall \sigma \in V_4, \sigma(\theta_j) = \pm\theta_j$. The θ_j arise from Lagrange resolvents for the three quadratic subextensions of L^{V_4} in L . They behave like $\sqrt{2}, \sqrt{3}, \sqrt{6}$ in $\mathbb{Q}(\sqrt{2}, \sqrt{3})$. Each $t_i = \theta_i^2$ is fixed by V_4 and are permuted by $S_4/V_4 \cong S_3$. They are roots of **cubic resolvent** of $f(x)$:

$$(t - t_1)(t - t_2)(t - t_3) = t^3 + s_1 t^2 + s_2 t + s_3$$

which has coefficients in $(L^{V_4})^{S_3} = L^{S_4} = K$. To find roots x_1, x_2, x_3, x_4 of $f(x)$:

- Solve cubic resolvent to find t_1, t_2, t_3 .
- Set $\theta_j = \pm\sqrt{t_j}$ where signs are chosen so that $\theta_1\theta_2\theta_3 = (e_1^3 - 4e_1e_2 + 8e_3)/8$.
- Solve the linear system

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = e_1 \\ x_1 + x_2 - x_3 + x_4 = 2\theta_1 \\ x_1 - x_2 + x_3 - x_4 = 2\theta_2 \\ x_1 - x_2 - x_3 + x_4 = 2\theta_3 \end{cases} \implies \begin{cases} x_1 = e_1/4 + (\theta_1 + \theta_2 + \theta_3)/2 \\ x_2 = e_1/4 + (\theta_1 - \theta_2 - \theta_3)/4 \\ x_3 = e_1/4 + (-\theta_1 + \theta_2 - \theta_3)/2 \\ x_4 = e_1/4 + (-\theta_1 - \theta_2 + \theta_3)/2 \end{cases}$$

Remark. In practice, perform shift to kill x^3 coefficient to obtain **reduced quartic**:

$$f(x - a/4) = x^4 + px^2 + qx + r$$

- Cubic resolvent is (*memorise*)

$$t^3 + 2pt^2 + (p^2 - 4r)t - q^2$$

- Choose $\theta_1, \theta_2, \theta_3$ such that (*memorise*)

$$\theta_1\theta_2\theta_3 = -q$$

- Roots of $f(x - a/4)$ are (*memorise*)

$$x_1 = \frac{1}{2}(\theta_1 + \theta_2 + \theta_3),$$

$$x_2 = \frac{1}{2}(\theta_1 - \theta_2 - \theta_3),$$

$$x_3 = \frac{1}{2}(-\theta_1 + \theta_2 - \theta_3),$$

$$x_4 = \frac{1}{2}(-\theta_1 - \theta_2 + \theta_3)$$

- Recover roots of $f(x)$ by subtracting $a/4$.

Example. Find all complex roots of $f(x) = x^4 + 6x^3 + 18x^2 + 30x + 25$.

- Eliminate x^3 term:

$$f(x - 6/4) = x^4 + \frac{9}{2}x^2 + 3x + \frac{85}{16}$$

- $p = 9/2, q = 3, r = 85/16$, so cubic resolvent is

$$t^3 + 2pt^2 + (p^2 - 4r)t - q^2 = t^3 + 9t^2 - t - 9 = (t - 1)(t + 1)(t + 9)$$

So roots are $t_1 = 1, t_2 = -1, t_3 = -9$. Set $\theta_1 = \sqrt{t_1} = 1, \theta_2 = \sqrt{t_2} = i, \theta_3 = \pm\sqrt{t_3} = \pm 3i$ so that $\theta_1\theta_2\theta_3 = -q = -3$, i.e. $\theta_3 = 3i$.

- So roots of $f(x - 3/2)$ are

$$x_1 = \frac{1}{2}(\theta_1 + \theta_2 + \theta_3) = \frac{1}{2}(1 + 4i),$$

$$x_2 = \frac{1}{2}(\theta_1 - \theta_2 - \theta_3) = \frac{1}{2}(1 - 4i),$$

$$x_3 = \frac{1}{2}(-\theta_1 + \theta_3 - \theta_2) = \frac{1}{2}(-1 - 2i),$$

$$x_4 = \frac{1}{2}(-\theta_1 - \theta_2 + \theta_3) = \frac{1}{2}(-1 + 2i)$$

- So roots of $f(x)$ are $-1 \pm 2i, -2 \pm i$.

Example (Galois groups of quartic polynomials).

- Let $\text{char}(K) \neq 2, 3, f(x) = x^4 + ax^3 + bx^2 + cx + d \in K[x]$. Galois group is $G_f = \text{Gal}(L/K)$ where L is splitting field for $f(x)$ over K , and G_f is subgroup of S_4 .

- Assume that $f(x)$ irreducible in $K[x]$. It can be shown there are five possible isomorphism classes of Galois groups: $S_4, A_4, V_4, \mathbb{Z}/4$ or D_4 .
- Let $R(t) \in K[t]$ be cubic resolvent of $f(x)$ with roots $t_1 = \theta_1^2, t_2 = \theta_2^2, t_3 = \theta_3^2$. Let M be splitting field of $R(t)$ over K , so

$$K \subset K(t_1, t_2, t_3) \subset M \subset L = M(\theta_1, \theta_2, \theta_3)$$

Theorem. Let $f(x) \in K[x]$ irreducible and have irreducible cubic resolvent $R(t) \in K[t]$ with roots $t_1 = \theta_1^2, t_2 = \theta_2^2, t_3 = \theta_3^2$. Let L be splitting field of $f(x)$ over K (so $G_f = \text{Gal}(L/K)$) and let M be splitting field of $R(t)$ over K (so $G_R = \text{Gal}(M/K)$).

- If $\Delta_R \in K^{\times 2}$ (i.e. $G_R \cong A_3$ and $[M : K] = 3$), then $G_f \cong A_4$.
- If $\Delta_R \in K^\times - K^{\times 2}$ (i.e. $G_R \cong S_3$ and $[M : K] = 6$), then $G_f \cong S_4$.

Proof.

- Sufficient to prove $[L : M] = 4$ since then $[L : K] = 12$ or 24 by Tower Law.
- Show M does not contain θ_1, θ_2 or θ_3 .
 - Suppose it does, so WLOG $\theta_1 \in M$. $\text{Gal}(M/K) \cong A_3$ or S_3 , so must be order 3 element $\sigma \in \text{Gal}(M/K)$. $\sigma(\theta_1)$ and $\sigma^2(\theta_1)$ are the other two roots θ_2 and θ_3 since $R(t)$ is irreducible and $\theta_1, \theta_2, \theta_3 \in M$. But this implies $M = L$ so $[L : K] = 3$ or 6 , but $4 \mid [L : K]$ since L contains roots of irreducible quartic.
- $M(\theta_1)/M$ is degree 2. Assume $\theta_2 \in M(\theta_1)$. $\text{Gal}(M(\theta_1)/M) = \{\text{id}, \tau\}$ for some $\tau : \theta_1 \mapsto -\theta_1$. Also $\theta_2^2 \in M$ so $\tau(\theta_2) = \pm\theta_2$.
 - If $\tau(\theta_2) = \theta_2$, then $\theta_2 \in M$: contradiction.
 - If $\tau(\theta_2) = -\theta_2$, then $\tau(\theta_1\theta_2) = (-\theta_1)(-\theta_2) = \theta_1\theta_2$ hence $\theta_1\theta_2 \in M$. But $\theta_1\theta_2\theta_3 \in K$ and $\theta_1\theta_2 \neq 0$ since $R(t)$ irreducible. But then $\theta_3 \in M$: contradiction.
- Hence $[M(\theta_1, \theta_2) : M] \geq 4$, and $\theta_1\theta_2\theta_3 \in M$ so $L = M(\theta_1, \theta_2)$ and $[L : M] = 4$.

□

Example.

- If $f(x) \in K[x]$ but cubic resolvent $R(t) \in K[t]$ is reducible, it is possible that all roots $t_1 = \theta_1^2, t_2 = \theta_2^2, t_3 = \theta_3^2$ are in K . Then $M = K$ and $L = K(\theta_1, \theta_2, \theta_3)$. Since $\theta_1\theta_2\theta_3 \in K$, L/K is obtained by adjoining only two square roots to K . Since $f(x)$ irreducible of degree 4, we have $[L : K] \geq 4$, hence only option is biquadratic extension $G_f = \text{Gal}(L/K) = V_4 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$.
- If only one root t_1, t_2, t_3 is in K (say it is t_1):
 - M is splitting field of irreducible quadratic over K . Hence $M = K(\sqrt{d})$ for some $d \in K^\times - K^{\times 2}$ and $\text{Gal}(M/K) = \{\text{id}, \varphi\} \cong \mathbb{Z}/2$ where $\varphi(\sqrt{d}) = -\sqrt{d}$.
 - We have

$$K \subset M = K(\sqrt{d}) = K(\alpha, \bar{\alpha}) \subset L = M(\sqrt{\alpha}, \sqrt{\bar{\alpha}})$$

where α and $\bar{\alpha} = \varphi(\alpha)$ are conjugate elements in $M^\times - M^{\times 2}$ (corresponding to t_2 and t_3).

- In this case, L/K is normal extension, since if $\alpha, \bar{\alpha}$ are roots of $x^2 + ax + b \in K[x]$, then $\pm\sqrt{\alpha}, \pm\sqrt{\bar{\alpha}}$ are roots of $x^4 + ax^2 + b \in K[x]$. So L is splitting field of $x^4 + ax^2 + b$ over K . For above tower of fields, we have Galois groups

$$\{\text{id}\} \subset \text{Gal}(L/M) = H \subset \text{Gal}(L/K) = G$$

and $G/H \cong \text{Gal}(M/K) = \{\text{id}, \varphi\} \cong \mathbb{Z}/2$.

Theorem. Let $M = K(\sqrt{d})$, $d \notin K^{\times 2}$, $\text{Gal}(M/K) = \{\text{id}, \varphi\}$. Let $\alpha, \bar{\alpha} = \varphi(\alpha) \in M^{\times} - M^{\times 2}$, and let $L = M(\sqrt{\alpha}, \sqrt{\bar{\alpha}})$, $G = \text{Gal}(L/K)$.

- If $\alpha\bar{\alpha} \in K^{\times 2}$, then $[L : K] = 4$ and $G \cong \mathbb{Z}/2 \times \mathbb{Z}/2$.
- If $\alpha\bar{\alpha} \in M^{\times 2} - K^{\times 2}$ then $[L : K] = 4$ and $G \cong \mathbb{Z}/4$.
- If $\alpha\bar{\alpha} \notin M^{\times 2}$, then $[L : K] = 8$ and $G \cong D_4$.

Note. In the case that $C := \alpha\bar{\alpha} \notin M^{\times 2}$ and so $G \cong D_4$:

- We have $\text{Gal}(M/K) = \{\text{id}, \varphi\}$, $\varphi : \sqrt{d} \mapsto -\sqrt{d}$.
- There are two lifts of φ to L :

$$\tau : (\sqrt{d}, \sqrt{C}, \sqrt{\alpha}) \mapsto (-\sqrt{d}, \sqrt{C}, \sqrt{\alpha}),$$

$$\sigma : (\sqrt{d}, \sqrt{C}, \sqrt{\alpha}) \mapsto (-\sqrt{d}, -\sqrt{C}, \sqrt{\alpha})$$

(so $\tau(\sqrt{\alpha}) = \sqrt{\alpha}$, $\sigma(\sqrt{\alpha}) = -\sqrt{\alpha}$)

- Then $G = \text{Gal}(L/K) = \langle \tau, \sigma \mid \tau^2 = \sigma^4 = e, \tau\sigma = \sigma^3\tau \rangle$.

8.4. A criterion for solvability by radicals

Note. Assume all fields in this section have characteristic 0.

Definition. L/K is **radical extension** if there is tower of field extensions

$$K = K_0 \subset \dots \subset K_m = L$$

where for each $1 \leq i \leq m$, $K_i = K_{i-1}(\sqrt[n_i]{\alpha_i})$ with $\alpha_i \in K_{i-1}$ and $n_i \in \mathbb{N}$.

Example. Let $\alpha = \sqrt[3]{2 + \sqrt[5]{3 - \sqrt{7}}}$. We have

$$K_0 = \mathbb{Q} \subset K_1 = \mathbb{Q}(\sqrt{7}) \subset K_2 = K_1\left(\sqrt[5]{3 - \sqrt{7}}\right) \subset K_3 = K_2(\alpha)$$

Definition. $f(x) \in K[x]$ is **solvable in radicals** over K if there is a radical extension L of K containing at least one root of $f(x)$.

Lemma. If $f(x)$ irreducible and solvable in radicals, then all its roots belong to the radical field extension L .

Definition. A finite group G is **solvable (soluble)** if there exists decreasing sequence of subgroups

$$G = G_0 \supset \dots \supset G_m = \{\text{id}\}$$

where for each $1 \leq i \leq m$, G_i is normal subgroup of G_{i-1} and G_{i-1}/G_i is cyclic.

Lemma (Properties of solvable groups).

- Every subgroup of finite solvable group is solvable.
- Abelian groups are solvable.
- S_n is solvable iff $n \leq 4$.

- Let G finite group with normal subgroup H . Then G is solvable iff both H and G/H are solvable.

Theorem (Galois' Theorem: Criterion for solvability in radicals). Let $f(x) \in K[x]$ irreducible. Then $f(x)$ is solvable in radicals over K iff Galois group G_f is solvable.

8.5. Polynomials not solvable by radicals

Lemma. A_n is generated by 3-cycles $(i j k)$.

Proof.

- $A_1 = A_2 = \{\text{id}\}$.
- For $n \geq 3$, any element in A_n is product of even number of transpositions.
- Combine pairs of transpositions as follows:
 - $(ij)(ij) = \text{id}$.
 - $(ij)(ik) = (ikj)$.
 - $(ij)(kl) = (ik)(jk)(jk)(kl) = (ijk)(jkl)$.

□

Theorem. For $n \geq 5$, A_n and S_n are not solvable.

Proof.

- Assume A_n solvable, so there is decreasing sequence of subgroups

$$A_n = G_0 \supset \dots \supset G_m = \{\text{id}\}$$

with G_i normal in G_{i-1} , G_{i-1}/G_i cyclic and so abelian. So we have canonical projection homomorphism $\pi : A_n \rightarrow Q = A_n/G_1$, Q is abelian and non-trivial.

- Let $g = (i_1 i_2 i_3) \in A_n$. There are $i_4, i_5 \in [n]$ (since $n \geq 5$) such that i_1, i_2, i_3, i_4, i_5 distinct. Let $g_1 = (i_1 i_2 i_4)$, $g_2 = (i_1 i_3 i_5)$, then $g_1 g_2 g_1^{-1} g_2^{-1} = g$.
- Since Q abelian, $\pi(g) = \pi(g_1) \pi(g_2) \pi(g_1)^{-1} \pi(g_2)^{-1} = \text{id}$.
- So π sends 3-cycles to id, and A_n is generated by 3-cycles, so $\pi(A_n) = \{\text{id}\}$ which is the trivial group: contradiction.

□

Theorem. Let $f(x) \in \mathbb{Q}[x]$ irreducible polynomial of degree 5 with exactly 3 real roots. Then $f(x)$ has Galois group $G_f \cong S_5$ (and so $f(x)$ is not solvable by radicals over \mathbb{Q}).