Contents

1. Combinatorial methods

Definition 1.1 Let G be an abelian group and $A, B \subseteq G$. The **sumset** of A and B is

$$A + B := \{a + b : a \in A, b \in B\}.$$

The **difference set** of A and B is

$$A - B := \{a - b : a \in A, b \in B\}.$$

Proposition 1.2 $\max\{|A|, |B|\} \leq |A + B| \leq |A| \cdot |B|$.

Proof. Trivial. □

Example 1.3 Let $A = [n] = \{1, ..., n\}$. Then $A + A = \{2, ..., 2n\}$ so $|A + A| = 2|A| - 1$.

Lemma 1.4 Let $A \subseteq \mathbb{Z}$ be finite. Then $|A + A| \geq 2|A| - 1$ with equality iff A is an arithmetic progression.

Proof (Hints). Consider two sequences in $A + A$ which are strictly increasing and of the same length. □

Proof. Let $A = \{a_1, ..., a_n\}$ with $a_i < a_{i+1}$. Then $a_1 + a_1 < a_1 + a_2 < \cdots < a_1 + a_n < a_2 + a_n < \cdots < a_n + a_n$. Note this is not the only choice of increasing sequence that works, in particular, so does $a_1 + a_1 < a_1 + a_2 < a_2 + a_2 < a_2 + a_3 < a_2 + a_4 < \cdots < a_2 + a_n < a_3 + a_n < \cdots < a_n + a_n$. So when equality holds, all these sequences must be the same. In particular, $a_2 + a_i = a_1 + a_{i+1}$ for all i. □

Lemma 1.5 If $A, B \subseteq \mathbb{Z}$, then $|A + B| \geq |A| + |B| - 1$ with equality iff A and B are arithmetic progressions with the same step.

Proof (Hints). Similar to above, consider 4 sequences in $A + B$ which are strictly increasing and of the same length. □

Example 1.6 Let $A, B \subseteq \mathbb{Z}/p$ for p prime. If $|A| + |B| \geq p + 1$, then $A + B = \mathbb{Z}/p$.

Proof (Hints). Consider $A \cap (g - B)$ for $g \in \mathbb{Z}/p$. □

Proof. Note that $g \in A + B$ iff $A \cap (g - B) \neq \emptyset$ where $(g - B = \{g\} - B)$. Let $g \in \mathbb{Z}/p$, then use inclusion-exclusion on $|A \cap (g - B)|$ to conclude result. □

Theorem 1.7 (Cauchy-Davenport) Let p be prime, $A, B \subseteq \mathbb{Z}/p$ be non-empty. Then

$$|A + B| \geq \min\{p, |A| + |B| - 1\}.$$

Proof (Hints).

- Assume $|A| + |B| < p + 1$, and WLOG that $1 \leq |A| \leq |B|$ and $0 \in A$ (by translation).
- Induct on $|A|$.
- Let $a \in A$, find B' such that $0 \in B'$, $a \notin B'$ and $|B'| = |B|$ (use fact that p is prime).

- Apply induction with $A \cap B'$ and $A \cup B'$, while reasoning that $(A \cap B') + (A \cup B') \subseteq A + B'$.

□

Proof. Assume $|A| + |B| < p + 1$, and WLOG that $1 \le |A| \le |B|$ and $0 \in A$ (by translation). We use induction on $|A|$. $|A| = 1$ is trivial. Let $|A| \ge 2$ and let $0 \ne a \in A$. Then since p is prime, $\{a, 2a, ..., pa\} = \mathbb{Z}/p$. There exists $m \ge 0$ such that $ma \in B$ but $(m+1)a \notin B$ (why?). Let $B' = B - ma$, so $0 \in B'$, $a \notin B'$ and $|B'| = |B|$.

Now $1 \le |A \cap B'| < |A|$ (why?) so the inductive hypothesis applies to $A \cap B'$ and $A \cup B'$. Since $(A \cap B') + (A \cup B') \subseteq A + B'$ (why?), we have $|A + B| = |A + B'| \ge |(A \cap B') + (A \cup B')| \ge |A \cap B'| + |A \cup B'| - 1 = |A| + |B| - 1$. □

Example 1.8 Cauchy-Davenport does not hold general abelian groups (e.g. $\mathbb{Z}/n$ for n composite): for example, let $A = B = \{0, 2, 4\} \subseteq \mathbb{Z}/6$, then $A + B = \{0, 2, 4\}$ so $|A + B| = 3 < \min\{6, |A| + |B| - 1\}$.

Example 1.9 Fix a small prime p and let $V \subseteq \mathbb{F}_p^n$ be a subspace. Then $V + V = V$, so $|V + V| = |V|$. In fact, if $A \subseteq \mathbb{F}_p^n$ satisfies $|A + A| = |A|$, then A is an affine subspace (a coset of a subspace).

Proof. If $0 \in A$, then $A \subseteq A + A$, so $A = A + A$. General result follows by considering translation of A. □

Example 1.10 Let $A \subseteq \mathbb{F}_p^n$ satisfy $|A + A| \le \frac{3}{2}\,|A|$. Then there exists a subspace $V \subseteq \mathbb{F}_p^n$ such that $|V| \le \frac{3}{2}\,|A|$ and A is contained in a coset of V.

Proof. Exercise (sheet 1). □

Definition 1.11 Let $A, B \subseteq G$ be finite subsets of an abelian group G. The **Ruzsa distance** between A and B is

$$d(A, B) := \log \frac{|A - B|}{\sqrt{|A| \cdot |B|}}.$$

Lemma 1.12 (Ruzsa Triangle Inequality) Let $A, B, C \subseteq G$ be finite. Then

$$d(A, C) \le d(A, B) + d(B, C).$$

Proof (Hints). Consider a certain map from $B \times (A - C)$ to $(A - B) \times (B - C)$. □

Proof. Note that $|B|\,|A - C| \le |A - B|\,|B - C|$. Indeed, writing each $d \in A - C$ as $d = a_d - c_d$ with $a_d \in A$, $c_d \in C$, the map $\varphi : B \times (A - C) \to (A - B) \times (B - C)$, $\varphi(b, d) = (a_d - b, b - c_d)$ is injective (why?). The triangle inequality now follows from definition of Ruzsa distance. □

Definition 1.13 The **doubling constant** of finite $A \subseteq G$ is $\sigma(A) := |A + A|/|A|$.

Definition 1.14 The **difference constant** of finite $A \subseteq G$ is $\delta(A) := |A - A|/|A|$.

Remark 1.15 The Ruzsa triangle inequality shows that

$$\log \delta(A) = d(A, A) \leq d(A, -A) + d(-A, A) = 2 \log \sigma(A).$$

So $\delta(A) \leq \sigma(A)^2$, i.e. $|A - A| \leq |A + A|^2/|A|$.

Notation 1.16 Let $A \subseteq G$, $\ell, m \in \mathbb{N}_0$. Then

$$\ell A + mA := \underbrace{A + \cdots + A}_{\ell \text{ times}} \underbrace{-A - \cdots - A}_{m \text{ times}}$$

This is referred to as the **iterated sum and difference set**.

Theorem 1.17 (Plunnecke's Inequality) Let $A, B \subseteq G$ be finite and $|A + B| \leq K|A|$ for some $K \geq 1$. Then $\forall \ell, m \in \mathbb{N}_0$,

$$|\ell B - mB| \leq K^{\ell+m}|A|.$$

Proof (Hints).

- Let $A' \subseteq A$ minimise $|A' + B|/|A'|$ with value K'.
- Show that for every finite $C \subseteq G$, $|A' + B + C| \leq K'|A + C|$ by induction on $|C|$ (note two sets need to be written as disjoint unions here).
- Show that $\forall m \in \mathbb{N}_0, |A' + mB| \leq (K')^m|A'|$ by induction.
- Use Ruzsa triangle inequality to conclude result.

□

Proof. Choose $\emptyset \neq A' \subseteq A$ which minimises $|A' + B|/|A'|$. Let the minimum value by K'. Then $|A' + B| = K'|A'|$, $K' \leq K$ and $\forall A'' \subseteq A$, $|A'' + B| \geq K'|A''|$.

We claim that for every finite $C \subseteq G$, $|A' + B + C| \leq K'|A' + C|$:

Use induction on $|C|$. $|C| = 1$ is true by definition of K'. Let claim be true for C, consider $C' = C \cup \{x\}$ for $x \notin C$. $A' + B + C' = (A' + B + C) \cup ((A' + B + x) - (D + B + x))$, where $D = \{a \in A' : a + B + x \subseteq A' + B + C\}$. By definition of K', $|D + B| \geq K'|D|$. Hence,

$$\begin{aligned}
|A' + B + C| &\leq |A' + B + C| + |A' + B + x| - |D + B + x| \\
&\leq K'|A' + C| + K'|A'| - K'|D| \\
&= K'(|A' + C| + |A'| - |D|).
\end{aligned}$$

Applying this argument a second time, write $A' + C' = (A' + C) \cup ((A' + x) - (E + x))$, where $E = \{a \in A' : a + x \in A' + C\} \subseteq D$. Finally,

$$\begin{aligned}
|A' + C'| &= |A' + C| + |A' + x| - |E + x| \\
&\geq |A' + C| + |A'| - |D|.
\end{aligned}$$

This proves the claim.

We now show that $\forall m \in \mathbb{N}_0$, $|A' + mB| \leq (K')^m|A'|$ by induction: $m = 0$ is trivial, $m = 1$ is true by assumption. Suppose it is true for $m - 1 \geq 1$. By the claim with $C = (m - 1)B$, we have

$$|A' + mB| = |A' + B + (m - 1)B| \leq K'|A' + (m - 1)B| \leq (K')^m|A'|.$$

As in the proof of Ruzsa's triangle inequality, $\forall \ell, m \in \mathbb{N}_0$,

$$\begin{aligned} |A'|\ |\ell B - mB| &\le |A' + \ell B|\ |A' + mB| \\ &\le (K')^{\ell}|A'|(K')^{m}|A'| \\ &= (K')^{\ell+m}|A'|^2. \end{aligned}$$

□

Theorem 1.18 (Freiman-Ruzsa) Let $A \subseteq \mathbb{F}_p^n$ and $|A + A| \le K|A|$. Then A is contained in a subspace $H \subseteq \mathbb{F}_p^n$ with $|H| \le K^2 p^{K^4}|A|$.

Proof (Hints).

- Let $X \subseteq 2A - A$ be of maximal size such that all $x + A$, $x \in X$, are disjoint.
- Use Plunnecke's Inequality to obtain an upper bound on $|X||A|$.
- Show that $\forall \ell \ge 2$, $\ell A - A \subseteq (\ell - 1)X + A - A$ by induction.
- Let H be subgroup generated by A. By writing H as an infinite union, show that $H \subseteq Y + A - A$, where Y is subgroup generated by X.
- Find an upper bound for $|Y|$, conclude using Plunnecke's Inequality.

□

Proof. Choose maximal $X \subseteq 2A - A$ such that the translates $x + A$ with $x \in X$ are disjoint. Such an X cannot be too large: $\forall x \in X$, $x + A \subseteq 3A - A$, so by Plunnecke's Inequality, since $|3A - A| \le K^4|A|$,

$$|X||A| = \left|\bigcup_{x \in X}(x + A)\right| \le |3A - A| \le K^4|A|.$$

Hence $|X| \le K^4$. We next show that $2A - A \subseteq X + A - A$. Indeed, if, $y \in 2A - A$ and $y \notin X$, then by maximality of X, then $(y + A) \cap (x + A) \ne \emptyset$ for some $x \in X$. If $y \in X$, then $y \in X + A - A$. It follows from above, by induction, that $\forall \ell \ge 2$, $\ell A - A \subseteq (\ell - 1)X + A - A$:

$$\begin{aligned} \ell A - A &= A + (\ell - 1)A - A \\ &\subseteq (\ell - 2)X + 2A - A \\ &\subseteq (\ell - 2)X + X + A - A \\ &= (\ell - 1)X + A - A. \end{aligned}$$

Now, let $H \subseteq \mathbb{F}_p^n$ be the subgroup generated by A:

$$H = \bigcup_{\ell \ge 1}(\ell A - A) \subseteq Y + A - A$$

where $Y \subseteq \mathbb{F}_p^n$ is the subgroup generated by X. Every element of Y can be written as a sum of $|X|$ elements of X with coefficients in $\{0, ..., p - 1\}$. Hence, $|Y| \le p^{|X|} \le p^{K^4}$. Finaly, $|H| \le |Y||A - A| \le p^{K^4}K^2|A|$ by Plunnecke's Inequality/Ruzsa Triangle Inequality. □

Example 1.19 Let $A = V \cup R$, where $V \subseteq \mathbb{F}_p^n$ is a subspace with $\dim(V) = d = n/K$ satisfying $K \ll d \ll n - K$, and R consists of $K - 1$ linearly independent vectors not in V. Then $|A| = |V \cup R| = |V| + |R| = p^{n/K} + K - 1 \approx p^{n/K} = |V|$.

Now $|A + A| = |(V \cup R) + (V \cup R)| = |V \cup (V + R) \cup 2R| \approx K|V| \approx K|A|$ (since $V \cup (V + R)$ gives K cosets of V). But any subspace $H \subseteq \mathbb{F}_p^n$ containing A must have size at least $p^{n/K+(K-1)} \approx |V|p^K$. Hence, the exponential dependence on K in Freiman-Ruzsa is necessary.

Theorem 1.20 (Polynomial Freiman-Ruzsa Theorem) Let $A \subseteq \mathbb{F}_p^n$ be such that $|A + A| \leq K|A|$. Then there exists a subspace $H \subseteq \mathbb{F}_p^n$ of size at most $C_1(K)|A|$ such that for some $x \in \mathbb{F}_p^n$,

$$|A \cap (x + H)| \geq \frac{|A|}{C_2(K)},$$

where $C_1(K)$ and $C_2(K)$ are polynomial in K.

Proof. Very difficult (took Green, Gowers and Tao to prove it). □

Definition 1.21 Given $A, B \subseteq G$ for an abelian group G, the **additive energy** between A and B is

$$E(A, B) := |\{(a, a', b, b') \in A \times A \times B \times B : a + b = a' + b'\}|.$$

Additive quadruples (a, a', b, b') are those such that $a + b = a' + b'$. Write $E(A)$ for $E(A, A)$.

Example 1.22 Let $V \subseteq \mathbb{F}_p^n$ be a subspace. Then $E(V) = |V|^3$. On the other hand, if $A \subseteq \mathbb{Z}/p$ is chosen at random from $\mathbb{Z}/p$ (where each $a \in \mathbb{Z}/p$ is included with probability $\alpha > 0$), with high probability, $E(A) = \alpha^4 p^3 = \alpha|A|^3$.

Definition 1.23 For $A, B \subseteq G$, the **representation function** is $r_{A+B}(x) := |\{(a, b) \in A \times B : a + b = x\}| = |A \cap (x - B)|$.

Lemma 1.24 Let $\emptyset \neq A, B \subseteq G$ for an abelian group G. Then

$$E(A, B) \geq \frac{|A|^2|B|^2}{|A \pm B|}.$$

Proof (Hints).

- Show that using Cauchy-Schwarz that

$$E(A, B) = \sum_{x \in G} r_{A+B}(x)^2 \geq \frac{\left(\sum_{x \in G} r_{A+B}(x)\right)^2}{|A + B|}.$$

- By using indicator functions, show that $\sum_{x \in G} r_{A+B}(x) = |A||B|$.

□

Proof. Observe that

$$
\begin{aligned}
E(A,B) &= \left|\{(a,a',b,b') \in A^2 \times B^2 : a+b = a'+b'\}\right| \\
&= \left|\bigcup_{x\in G} \{(a,a',b,b') \in A^2 \times B^2 : a+b = x \text{ and } a'+b' = x\}\right| \\
&= \bigcup_{x\in G} \left|\{(a,a',b,b') \in A^2 \times B^2 : a+b = x \text{ and } a'+b' = x\}\right| \\
&= \sum_{x\in G} r_{A+B}(x)^2 \\
&= \sum_{x\in A+B} r_{A+B}(x)^2 \\
&\geq \frac{\left(\sum_{x\in A+B} r_{A+B}(x)\right)^2}{|A+B|} \qquad \text{by Cauchy-Schwarz}
\end{aligned}
$$

But now

$$
\begin{aligned}
\sum_{x\in G} r_{A+B}(x) = \sum_{x\in G} |A \cap (x-B)| &= \sum_{x\in G}\sum_{y\in G} \mathbb{1}_A(y)\mathbb{1}_{x-B}(y) \\
&= \sum_{x\in G}\sum_{y\in G} \mathbb{1}_A(y)\mathbb{1}_B(x-y) = |A||B|.
\end{aligned}
$$

Note that the same argument works for $|A-B|$. □

Corollary 1.25 If $|A+A| \leq K|A|$, then $E(A) \geq \frac{|A|^4}{|A+A|} \geq \frac{|A|^3}{K}$. So if A has small doubling constant, then it has large additive energy.

Proof (Hints). Trivial. □

Proof. Trivial. □

Example 1.26 The converse of the above lemma does not hold: e.g. let G be a (class of) abelian group(s). Then there exist constants $\theta, \eta > 0$ such that for all n large enough, there exists $A \subseteq G$ with $|A| \geq n$ satisfying $E(A) \geq \eta|A|^3$, and $|A+A| \geq \theta|A|^2$.

Definition 1.27 Given $A \subseteq G$ and $\gamma > 0$, let $P_\gamma := \{x \in G : |A \cap (x+A)| \geq \gamma|A|\}$ be the set of **γ-popular differences** of A.

Lemma 1.28 Let $A \subseteq G$ be finite such that $E(A) = \eta|A|^3$ for some $\eta > 0$. Then $\forall c > 0$, there is a subset $X \subseteq A$ with $|X| \geq \frac{\eta}{3}|A|$ such that for all $(16c)$-proportion of pairs $(a,b) \in X^2$, $a - b \in P_{c\eta}$.

Proof. We use a technique called "dependent random choice". Let $U = \left\{x \in G : |A \cap (x+A)| \leq \frac{1}{2}\eta|A|\right\}$. Then

$$
\begin{aligned}
\sum_{x\in U} |A \cap (x+A)|^2 &\leq \frac{1}{2}\eta|A| \sum_{x\in G} |A \cap (x+A)| \\
&= \frac{1}{2}\eta|A|^3 = \frac{1}{2}E(A).
\end{aligned}
$$

For $0 \le i \le \lceil \log_2 \eta^{-1} \rceil$, let $Q_i = \{x \in G : |A|/2^{i+1} < |A \cap (x+A)| \le |A|/2^i\}$ and set $\delta_i = \eta^{-1} 2^{-2i}$. Then

$$\begin{aligned}
\sum_{i=0}^{\lceil \log_2 \eta^{-1} \rceil} \delta_i |Q_i| &= \sum_i \frac{|Q_i|}{\eta 2^{2i}} \\
&= \frac{1}{\eta |A|^2} \sum_i \frac{|A|^2}{2^{2i}} |Q_i| \\
&= \frac{1}{\eta |A|^2} \sum_i \frac{|A|^2}{2^{2i}} \sum_{x \notin U} \mathbb{1}_{\{|A|/2^{i+1} < |A \cap (x+A)| \le |A|/2^i\}} \\
&\ge \frac{1}{\eta |A|^2} \sum_{x \notin U} |A \cap (x+A)|^2 \\
&\ge \frac{1}{\eta |A|^2} \cdot \frac{1}{2} E(A) = \frac{1}{2} |A|.
\end{aligned}$$

Let $S = \{(a,b) \in A^2 : a - b \notin P_{c\eta}\}$. Now

$$\begin{aligned}
\sum_i \sum_{(a,b) \in S} |(A-a) \cap (A-b) \cap Q_i| &\le \sum_{(a,b) \in S} |(A-a) \cap (A-b)| \\
&= \sum_{(a,b) \in S} |A \cap (a-b+A)| \\
&\le \sum_{(a,b) \in S} c\eta |A| \quad \text{by definition of } S \\
&= |S| c\eta |A| \\
&\le c\eta |A|^3 = 2c\eta |A|^2 \cdot \frac{1}{2} |A| \\
&\le 2c\eta |A|^2 \sum_i \delta_i |Q_i| \quad \text{by above inequality.}
\end{aligned}$$

Hence $\exists i_0$ such that

$$\sum_{(a,b) \in S} \left| (A-a) \cap (A-b) \cap Q_{i_0} \right| \le 2c\eta |A|^2 \delta_{i_0} \left| Q_{i_0} \right|.$$

Let $Q = Q_{i_0}$, $\delta = \delta_{i_0}$, $\lambda = 2^{-i_0}$, so that

$$\sum_{(a,b) \in S} |(A-a) \cap (A-b) \cap Q| \le 2c\eta |A|^2 \delta |Q|.$$

Given $x \in G$, let $X(x) = A \cap (x+A)$. Then

$$\mathbb{E}_{x \in Q} |X(x)| = \frac{1}{|Q|} \sum_{x \in Q} |A \cap (x+A)| \ge \frac{1}{2} \lambda |A|.$$

Define $T(x) = \{(a,b) \in X(x)^2 : a - b \in P^{c\eta}\}$. Then

$$
\begin{aligned}
\mathbb{E}_{x\in Q}|T(x)| &= \mathbb{E}_{x\in Q}\big|\{(a,b)\in(A\cap(x+A))^2 : a-b\notin P_{c\eta}\}\big| \\
&= \frac{1}{|Q|}\sum_{x\in Q}|\{(a,b)\in S : x\in(A-a)\cap(A-b)\}| \\
&= \frac{1}{|Q|}\sum_{(a,b)\in S}|(A-a)\cap(A-b)\cap Q| \\
&\le \frac{1}{|Q|}2c\eta|A|^2\delta|Q| = 2c\eta\delta|A|^2 = 2c\lambda^2|A|^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbb{E}_{x\in Q}(|X(x)|^2-(16c)^{-1}|T(x)|) &\ge \big(\mathbb{E}_{x\in Q}|X(x)|\big)^2-(16c)^{-1}\mathbb{E}_{x\in Q}|T(x)| \text{ by Cauchy-Schwarz} \\
&\ge \left(\frac{\lambda}{2}\right)^2|A|^2-(16c)^{-1}2c\lambda^2|A|^2 \\
&= \left(\frac{\lambda^2}{4}-\frac{\lambda^2}{8}\right)|A|^2 = \frac{\lambda^2}{8}|A|^2.
\end{aligned}
$$

So $\exists x\in Q$ such that $|X(x)|^2\ge\frac{\lambda^2}{8}|A|^2$, so $|X|\ge\frac{\lambda}{\sqrt{8}}|A|\ge\frac{\eta}{3}|A|$ and $|T(x)|\le 16c|X|^2$. □

Theorem 1.29 (Balog-Szemerédi-Gowers, Schoen) Let $A\subseteq G$ be finite such that $E(A)\ge\eta|A|^3$ for some $\eta>0$. Then there exists $A'\subseteq A$ with $|A'|\ge c_1(\eta)|A|$ such that $|A'+A'|\le|A|/c_2(\eta)$, where $c_1(\eta)$ and $c_2(\eta)$ are both polynomial in η.

Proof. The idea is to find $A'\subseteq A$ such that $\forall a,b\in A'$, $a-b$ has many representations as $(a_1-a_2)+(a_3-a_4)$ with each $a_i\in A$. Apply the above lemma with $c=2^{-7}$ to obtain $X\subseteq A$ with $|X|\ge\frac{\eta}{3}|A|$ such that for all but $\frac{1}{8}$ of pairs $(a,b)\in X^2$, $a-b\in P_{\eta/2^7}$. In particular, the bipartite graph $G=(X\sqcup X,\{(x,y)\in X\times X : x-y\in P_{\eta/2^7}\})$ has at least $\frac{7}{8}|X|^2$ edges.

Let $A'=\{x\in X:\deg_G(x)\ge\frac{3}{4}|X|\}$. Clearly $|A'|\ge|X|/8$. For any $a,b\in A'$, there are at least $|X|/2$ elements $y\in X$ such that $(a,y),(b,y)\in E(G)$ (so $a-y,b-y\in P_{\eta/2^7}$). Hence $a-b=(a-y)-(b-y)$ has at least

$$
\underbrace{\frac{\eta}{6}|A|}_{\text{choices for } y}\cdot\frac{\eta}{2^7}|A|\frac{\eta}{2^7}|A|\ge\frac{\eta^3}{2^{17}}|A|^3
$$

representations of the form $a_1-a_2-(a_3-a_4)$ with each $a_i\in A$. It follows that $\frac{\eta^3}{2^{17}}|A|^3|A'-A'|\le|A|^4$, hence $|A'-A'|\le 2^{17}\eta^{-3}|A|\le 2^{22}\eta^{-4}|A'|$, and so $|A'+A'|\le 2^{44}\eta^{-8}|A'|$. □

2. Fourier-analytic techniques

In this chapter, assume that G is a *finite* abelian group.

Definition 2.1 The group $\hat{G}$ of **characters** of G is the group of homomorphisms $\gamma: G\to\mathbb{C}^\times$. In fact, $\hat{G}$ is isomorphic to G.

Notation 2.2 Norm and inner product notation:

- Write

$$\|f\|_q = \|f\|_{L^q(G)} = (\mathbb{E}_{x\in G}|f(x)|^q)^{1/q},$$
$$\|\hat{f}\|_q = \|\hat{f}\|_{\ell^q(\widehat{G})} = (\sum_{\gamma\in\widehat{G}} |\hat{f}(\gamma)|^q)^{1/q},$$
$$\langle f, g\rangle_{L^2(G)} = \mathbb{E}_{x\in G} f(x)\overline{g(x)},$$
$$\langle f, g\rangle_{\ell^2(\widehat{G})} = \sum_{\gamma\in\widehat{G}} \hat{f}(\gamma)\overline{\hat{g}(\gamma)}$$

- If Fourier support of function is restricted to $\Lambda \subseteq \widehat{G}$, write $\|\hat{f}\|_{\ell^q(\Lambda)} = \left(\sum_{\gamma\in\Lambda} |\hat{f}(\gamma)|^q\right)^{1/q}$.

Notation 2.3 Asymptotic notation:

- Write $f(n) = O(g(n))$ if

$$\exists C > 0 : \forall n \in \mathbb{N}, \quad |f(n)| \le C|g(n)|.$$

- Write $f(n) = o(g(n))$ if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \ge N, |f(n)| \le \varepsilon|g(n)|,$$

  i.e. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.
- Write $f(n) = \Omega(g(n))$ if $g(n) = O(f(n))$.
- If the implied constant depends on a fixed parameter, this may be indicated by a subscript, e.g. $\exp(p + n^2) = O_p(\exp(n^2))$.

Theorem 2.4 (Hölder's Inequality) Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q}$, and $f \in L^p(G)$, $g \in L^q(G)$. Then

$$\|fg\|_1 \le \|f\|_p \|g\|_q.$$

Theorem 2.5 (Cauchy-Schwarz Inequality) For $f, g \in L^2(G)$, we have

$$\langle f, g\rangle_{L^2(G)} \le \|f\|_2 \|g\|_2.$$

Note this is a special case of Hölder's inequality with $p = q = 2$.

Theorem 2.6 (Young's Convolution Inequality) Let $p, q, r \in [1, \infty]$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $f \in L^p(G)$, $g \in L^q(G)$. Then

$$\|f * g\|_r \le \|f\|_p \|g\|_q.$$

Notation 2.7 $e(y)$ denotes the function $e^{2\pi i y}$.

Example 2.8

- Let $G = \mathbb{F}_p^n$, then for any $\gamma \in \widehat{G}$, we have a corresponding character $\gamma(x) = e((\gamma.x)/p)$.
- If $G = \mathbb{Z}/N$, then any $\gamma \in \widehat{G}$ has a corresponding character $\gamma(x) = e(\gamma x/N)$.

Notation 2.9 Given a non-empty $B \subseteq G$ and $g : B \to \mathbb{C}$, write $\mathbb{E}_{x \in B} g(x)$ for $\frac{1}{|B|} \sum_{x \in B} g(x)$. If $B = G$, we may simply write $\mathbb{E}$ instead of $\mathbb{E}_{x \in B}$.

Lemma 2.10 For all $\gamma \in \hat{G}$,

$$\mathbb{E}_{x \in G} \gamma(x) = \begin{cases} 1 & \text{if } \gamma = 1 \\ 0 & \text{otherwise} \end{cases}.$$

and for all $x \in G$,

$$\sum_{\gamma \in \widehat{G}} \gamma(x) = \begin{cases} |G| & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Proof (Hints).

- For $1 \neq \gamma \in \hat{G}$, consider $y \in G$ with $\gamma(y) \neq 1$.
- For $0 \neq x \in G$, by considering $G/\langle x \rangle$, show by contradiction that there is $\gamma \in \hat{G}$ with $\gamma(x) \neq 1$.

□

Proof. The first case for both equations is trivial. Let $1 \neq \gamma \in \hat{G}$. Then $\exists y \in G$ with $\gamma(y) \neq 1$. So

$$\begin{aligned} \gamma(y) \mathbb{E}_{z \in G} \gamma(z) &= \mathbb{E}_{z \in G} \gamma(y + z) \\ &= \mathbb{E}_{z' \in G} \gamma(z'). \end{aligned}$$

Hence $\mathbb{E}_{z \in G} \gamma(z) = 0$.

For second equation, given $0 \neq x \in G$, there exists $\gamma \in \hat{G}$ such that $\gamma(x) \neq 1$, since otherwise $\hat{G}$ would act trivially on $\langle x \rangle$, hence would also be the dual group for $G/\langle x \rangle$, a contradiction. □

Definition 2.11 Given $f : G \to \mathbb{C}$, define the **Fourier transform** of f to be

$$\begin{aligned} \hat{f} : \hat{G} &\to \mathbb{C}, \\ \gamma &\mapsto \mathbb{E}_{x \in G} f(x) \overline{\gamma(x)}. \end{aligned}$$

Proposition 2.12 (Fourier Inversion Formula) Let $f : G \to \mathbb{C}$. Then for all $x \in G$,

$$f(x) = \sum_{\gamma \in \widehat{G}} \hat{f}(\gamma) \gamma(x).$$

Proof (Hints). Straightforward. □

Proof. We have

$$\begin{aligned}\sum_{\gamma\in\widehat{G}}\hat{f}(\gamma)\gamma(x) &= \sum_{\gamma\in\widehat{G}}\mathbb{E}_{y\in G}f(y)\overline{\gamma(y)}\gamma(x)\\ &= \mathbb{E}_{y\in G}f(y)\sum_{\gamma\in\widehat{G}}\gamma(x-y)\\ &= f(x)\end{aligned}$$

by Lemma 2.10. □

Definition 2.13 For $A \subseteq G$, the **indicator** (or **characteristic**) function of A is

$$\begin{aligned}\mathbb{1}_A &: G \to \{0,1\},\\ x &\mapsto \begin{cases}1 & \text{if } x\in A\\ 0 & \text{if } x\notin A\end{cases}.\end{aligned}$$

Definition 2.14 $\hat{\mathbb{1}}_A(1) = \mathbb{E}_{x\in G}\mathbb{1}_A(x)\cdot 1 = |A|/|G|$ is the **density** of A in G. This is often denoted by α.

Definition 2.15 Given $\emptyset \neq A \subseteq G$, the **characteristic measure** $\mu_A : G \to [0,|G|]$ is defined by

$$\mu_A(x) \coloneqq \alpha^{-1}\mathbb{1}_A(x).$$

Note that $\mathbb{E}_{x\in G}\mu_A(x) = 1 = \hat{\mu}_A(1)$.

Definition 2.16 The **balanced function** $f_A : G \to [-1,1]$ of A is given by

$$f_A(x) = \mathbb{1}_A(x) - \alpha.$$

Note that $\mathbb{E}_{x\in G}f_A(x) = 0 = \hat{f}_A(1)$.

Example 2.17 Let $V \leq \mathbb{F}_p^n$ be a subspace. Then for $t \in \hat{\mathbb{F}}_p^n$,

$$\begin{aligned}\hat{\mathbb{1}}_V(t) &= \mathbb{E}_{x\in\mathbb{F}_p^n}\mathbb{1}_V(x)e(-x.t/p)\\ &= \frac{|V|}{p^n}\mathbb{1}_{V^\perp}(t).\end{aligned}$$

where $V^\perp = \{t \in \hat{\mathbb{F}}_p^n : x.t = 0 \quad \forall x \in V\}$ is the **annihilator** of V. Hence, $\hat{\mathbb{1}}_V = \mu_{V^\perp}$.

Example 2.18 Let $R \subseteq G$ be such that each $x \in G$ lies in R independently with probability $\frac{1}{2}$. Then with high probability,

$$\sup_{\gamma\neq 1}\left|\hat{\mathbb{1}}_R(\gamma)\right| = O\left(\sqrt{\frac{\log|G|}{|G|}}\right).$$

This follows from Chernoff's inequality.

Theorem 2.19 (Chernoff's Inequality) Given complex-valued independent random variables $X_1, ..., X_n$ with mean 0, for all $\theta > 0$, we have

$$\Pr\left[\left|\sum_{i=1}^n X_i\right| \geq \theta\sqrt{\sum_{i=1}^n \|X_i\|^2_{L^\infty(\Pr)}}\right] \leq 4\exp(-\theta^2/4).$$

Example 2.20 Let $Q = \left\{x \in \mathbb{F}_p^n : x.x = 0\right\}$ with $p > 2$. Then $|Q|/p^n = \frac{1}{p} + O\left(p^{-n/2}\right)$ and $\sup_{t\neq 0}\left|\widehat{\mathbb{1}}_Q(t)\right| = O\left(p^{-n/2}\right)$.

Lemma 2.21 (Plancherel's Identity) For all $f, g : G \to \mathbb{C}$,

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle.$$

Proof. Exercise. □

Corollary 2.22 (Parseval's Identity) For all $f, g : G \to \mathbb{C}$,

$$\|f\|^2_{L^2(G)} = \left\|\hat{f}\right\|^2_{\ell^2(\widehat{G})}.$$

Proof (Hints). Trivial from Plancherel's Identity. □

Proof. By Plancherel's Identity. □

Definition 2.23 Let $\rho > 0$ and $f : G \to \mathbb{C}$. The ***ρ*-large Fourier spectrum** of f is

$$\mathrm{Spec}_\rho(f) := \left\{\gamma \in \hat{G} : \left|\hat{f}(\gamma)\right| \geq \rho\|f\|_1\right\}.$$

Example 2.24 Let $A \subseteq G$, then $\|f\|_1 = \alpha = |A|/|G|$, so

$$\mathrm{Spec}_\rho(\mathbb{1}_A) = \left\{t \in \widehat{\mathbb{F}}_p^n : \left|\widehat{\mathbb{1}}_V(t)\right| \geq \rho\alpha\right\}.$$

In particular, if $V \leq \mathbb{F}_p^n$ is a subspace, then by Example 2.17, $\mathrm{Spec}_\rho(\mathbb{1}_V) = V^\perp$ for all $\rho \in (0, 1]$.

Lemma 2.25 For all $\rho > 0$,

$$\left|\mathrm{Spec}_\rho(f)\right| \leq \rho^{-2}\frac{\|f\|_2^2}{\|f\|_1^2}$$

In particular, if $f = \mathbb{1}_A$ for $A \subseteq G$, then $\|f\|_1 = \alpha = |A|/|G| = \|f\|_2^2$. So $\left|\mathrm{Spec}_\rho(\mathbb{1}_A)\right| \leq \rho^{-2}\alpha^{-1}$.

Proof (Hints). Use Parseval. □

Proof. By Parseval,

$$\begin{aligned}
\|f\|_2^2 = \left\|\hat{f}\right\|_2^2 = \sum_{\gamma\in\widehat{G}} \left|\hat{f}(\gamma)\right|^2 \\
&\geq \sum_{\gamma\in \mathrm{Spec}_\rho(f)} \left|\hat{f}(\gamma)\right|^2 \\
&\geq \left|\mathrm{Spec}_\rho(f)\right|(\rho\|f\|_1)^2.
\end{aligned}$$

□

Definition 2.26 The **convolution** of $f, g : \mathbb{G} \to \mathbb{C}$ is

$$\begin{aligned} f * g : G &\to \mathbb{C}, \\ x &\mapsto \mathbb{E}_{y \in G} f(y) g(x - y). \end{aligned}$$

Example 2.27 Given $A, B \subseteq G$,

$$\begin{aligned} (\mathbb{1}_A * \mathbb{1}_B)(x) &= \mathbb{E}_{y \in G} \mathbb{1}_A(y) \mathbb{1}_B(x - y) \\ &= \mathbb{E}_{y \in G} \mathbb{1}_A(y) \mathbb{1}_{x-B}(y) \\ &= \mathbb{E}_{y \in G} \mathbb{1}_{A \cap (x-B)}(y) \\ &= \frac{|A \cap (x - B)|}{|G|} = \frac{1}{|G|} r_{A+B}(x). \end{aligned}$$

In particular, $\mathrm{supp}(\mathbb{1}_A * \mathbb{1}_B) = A + B$.

Lemma 2.28 Given $f, g : G \to \mathbb{C}$,

$$\forall \gamma \in \hat{G}, \quad (\widehat{f * g})(\gamma) = \hat{f}(\gamma) \hat{g}(\gamma).$$

Proof (Hints). Straightforward. □

Proof. We have

$$\begin{aligned} (\widehat{f * g})(\gamma) &= \mathbb{E}_{x \in G} (f * g)(x) \overline{\gamma(x)} \\ &= \mathbb{E}_{x \in G} \mathbb{E}_{y \in G} f(y) g(x - y) \overline{\gamma(x)} \\ &= \mathbb{E}_{u \in G} \mathbb{E}_{y \in G} f(y) g(u) \overline{\gamma(u + y)} \quad (u = x - y) \\ &= \mathbb{E}_{u \in G} \mathbb{E}_{y \in G} f(y) g(u) \overline{\gamma(u) \gamma(y)} \\ &= \hat{f}(\gamma) \hat{g}(\gamma). \end{aligned}$$

□

Example 2.29 $\mathbb{E}_{x+y=z+w} f(x) f(y) \overline{f(z) f(w)} = \left\|\hat{f}\right\|_{\ell^4(\widehat{G})}^4$. In particular, $\left\|\hat{\mathbb{1}}_A\right\|_{\ell^4(\widehat{G})}^4 = E(A)/|G|^3$ for any $A \subseteq G$.

Theorem 2.30 (Bogolyubov's Lemma) Let $A \subseteq \mathbb{F}_p^n$ be of density α. Then there exists a subspace $V \leq \mathbb{F}_p^n$ with $\mathrm{codim}(V) \leq 2\alpha^{-2}$, such that $V \subseteq A + A - A - A$.

Proof (Hints).

- Let $g = \mathbb{1}_A * \mathbb{1}_A * \mathbb{1}_{-A} * \mathbb{1}_{-A}$, reason that if $g(x) > 0$ for all $x \in V$, then $V \subseteq 2A - 2A$.
- Let $S = \mathrm{Spec}_\rho(\mathbb{1}_A)$, with ρ for now unspecified.
- Show that $g(x) = \alpha^4 + \sum_{t \in S \setminus \{0\}} \left|\hat{\mathbb{1}}_A(t)\right|^4 e(x.t/p) + \sum_{t \notin S} \left|\hat{\mathbb{1}}_A(t)\right|^4 e(x.t/p)$.
- Find an appropriate subspace V from S, bound $g(x)$ from below in terms of ρ, and use this to determine a suitable value for ρ.

□

Proof. Observe $2A - 2A = \text{supp}(g)$ where $g = \mathbb{1}_A * \mathbb{1}_A * \mathbb{1}_{-A} * \mathbb{1}_{-A}$, so we want to find $V \le \mathbb{F}_p^n$ such that $g(x) > 0$ for all $x \in V$. Let $S = \text{Spec}_\rho(\mathbb{1}_A)$ with ρ a constant to be specified later, and let $V = \langle S \rangle^\perp$. By Lemma 2.25, $\text{codim}(V) = \dim\langle S \rangle \le |S| \le \rho^{-2}\alpha^{-1}$. Fix $x \in V$. Now

$$\begin{aligned} g(x) &= \sum_{t \in \hat{\mathbb{F}}_p^n} \hat{g}(t) e(x.t/p) \\ &= \sum_{t \in \hat{\mathbb{F}}_p^n} \left|\hat{\mathbb{1}}_A(t)\right|^4 e(x.t/p) \quad \text{by Lemma 2.28} \\ &= \alpha^4 + \sum_{t \neq 0} \left|\hat{\mathbb{1}}_A(t)\right|^4 e(x.t/p) \\ &= \alpha^4 + \sum_{t \in S \setminus \{0\}} \left|\hat{\mathbb{1}}_A(t)\right|^4 e(x.t/p) + \sum_{t \notin S} \left|\hat{\mathbb{1}}_A(t)\right|^4 e(x.t/p) \end{aligned}$$

Each term in the first sum is non-negative, since $\forall t \in S$, $x.t = 0$. The absolute value of the second sum is bounded above, by the triangle inequality, by

$$\begin{aligned} \sum_{t \notin S} \left|\hat{\mathbb{1}}_A(t)\right|^4 &\le \sup_{t \notin S} \left|\hat{\mathbb{1}}_A(t)\right|^2 \sum_{t \notin S} \left|\hat{\mathbb{1}}_A(t)\right|^2 \\ &\le \sup_{t \notin S} \left|\hat{\mathbb{1}}_A(t)\right|^2 \sum_{t \in \hat{\mathbb{F}}_p^n} \left|\hat{\mathbb{1}}_A(t)\right|^2 \\ &\le (\rho\alpha)^2 \|\mathbb{1}_A\|_2^2 = \rho^2\alpha^3 \end{aligned}$$

by Example 2.24 and Parseval. Note the second sum must be real since all other terms in the equation are. So we have $g(x) \ge \alpha^4 - \rho^2\alpha^3$. Thus, it is sufficient that $\rho^2\alpha^3 \le \frac{\alpha^4}{2}$, so set $\rho = \sqrt{a/2}$. Hence $g(x) > 0$ (in fact, $g(x) \ge \frac{\alpha^4}{2}$) for all $x \in V$, and $\text{codim}(V) \le 2\alpha^{-2}$. □

Example 2.31 The set $A = \left\{x \in \mathbb{F}_2^n : |x| \ge \frac{n}{2} + \frac{\sqrt{n}}{2}\right\}$ (where $|x|$ is number of 1s in x) has density $\ge \frac{1}{8}$ but there is no coset C of any subspace of codimension $\sqrt{n}$ such that $C \subseteq A + A$. Hence, the $2A - 2A$ part of Bogolyubov's lemma is necessary: $2A$ is not sufficient.

Lemma 2.32 Let $A \subseteq \mathbb{F}_p^n$ have density α with $\sup_{t \neq 0}\left|\hat{\mathbb{1}}_A(t)\right| \ge \rho\alpha$ for some $\rho > 0$. Then there exists a subspace $V \le \mathbb{F}_p^n$ with $\text{codim}(V) = 1$ and $x \in \mathbb{F}_p^n$ such that

$$|A \cap (x + V)| \ge \alpha\left(1 + \frac{\rho}{2}\right)|V|.$$

Proof (Hints).

- Let $V = \langle t \rangle^\perp$ for some suitable t (can determine later).
- Define $a_j = \frac{|A \cap (v_j + V)|}{|v_j + V|} - \alpha$ for each $j \in [p]$, where $x.v_j = j$.
- Show that $\hat{\mathbb{1}}_A(t) = \mathbb{E}_{j \in [p]} a_j e(-j/p)$.
- Show that $\mathbb{E}_{j \in [p]} a_j + |a_j| \ge \rho\alpha$.

□

Proof. Let $t \neq 0$ be such that $\left|\hat{\mathbb{1}}_A(t)\right| \geq \rho\alpha$ and let $V = \langle t \rangle^\perp$. Write $v_j + V = \left\{x \in \mathbb{F}_p^n : x.t = j\right\}$ for $j \in [p]$ for the p distinct cosets of V. Then

$$\begin{aligned}
\hat{\mathbb{1}}_A(t) = \hat{f}_A(t) &= \mathbb{E}_{x \in \mathbb{F}_p^n}(\mathbb{1}_A(x) - \alpha)e(-x.t/p) \\
&= \mathbb{E}_{j \in [p]}\mathbb{E}_{x \in v_j + V}(\mathbb{1}_A(x) - \alpha)e(-j/p) \\
&= \mathbb{E}_{j \in [p]}\left(\frac{|A \cap (v_j + V)|}{|v_j + V|} - \alpha\right)e(-j/p) \\
&=: \mathbb{E}_{j \in [p]} a_j e(-j/p).
\end{aligned}$$

By the triangle inequality, $\mathbb{E}_{j \in [p]}\left|a_j\right| \geq \rho\alpha$. Note that $\mathbb{E}_{j \in [p]} a_j = 0$. So $\mathbb{E}_{j \in [p]} a_j + \left|a_j\right| \geq \rho\alpha$, so $\exists j \in [p]$ such that $a_j + \left|a_j\right| \geq \rho\alpha$, hence $a_j \geq \rho\alpha/2$. So take $x = v_j$. □

Notation 2.33 Given $f, g, h : G \to \mathbb{C}$, write

$$T_3(f, g, h) = \mathbb{E}_{x, d \in G} f(x)g(x + d)h(x + 2d).$$

Notation 2.34 Given $A \subseteq G$, write $2 \cdot A = \{2a : a \in A\}$. Note this is not the same as $2A = A + A$.

Lemma 2.35 Let $p \geq 3$ and $A \subseteq \mathbb{F}_p^n$ be of density $\alpha > 0$, such that $\sup_{t \neq 0}\left|\hat{\mathbb{1}}_A(t)\right| \leq \varepsilon$. Then the number of 3-APs in A differs from $\alpha^3 (p^n)^2$ by at most $\varepsilon (p^n)^2$.

Proof (Hints).

- Express $T_3(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A)$ as an inner product of functions $\mathbb{F}_p^n \to \mathbb{C}$, rewrite as inner product of functions $\hat{\mathbb{F}}_p^n \to \mathbb{C}$.
- Find upper bound of the absolute value of a sub-sum of this inner product, using triangle inequality and Cauchy-Schwarz.

□

Proof. The number of 3-APs in A is $(p^n)^2$ multiplied by

$$\begin{aligned}
T_3(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) &= \mathbb{E}_{x,d}\mathbb{1}_A(x)\mathbb{1}_A(x + d)\mathbb{1}_A(x + 2d) \\
&= \mathbb{E}_{x,y}\mathbb{1}_A(x)\mathbb{1}_A(y)\mathbb{1}_A(2y - x) \\
&= \mathbb{E}_y\mathbb{1}_A(y)\mathbb{E}_x\mathbb{1}_A(x)\mathbb{1}_A(2y - x) \\
&= \mathbb{E}_y\mathbb{1}_A(y)(\mathbb{1}_A * \mathbb{1}_A)(2y) \\
&= \langle \mathbb{1}_{2 \cdot A}, \mathbb{1}_A * \mathbb{1}_A \rangle.
\end{aligned}$$

By Plancherel's Identity and Lemma 2.28, this is equal to

$$\begin{aligned}
\langle \hat{\mathbb{1}}_{2 \cdot A}, \hat{\mathbb{1}}_A^2 \rangle &= \sum_{t \in \hat{\mathbb{F}}_p^n} \hat{\mathbb{1}}_{2 \cdot A}(t)\overline{\hat{\mathbb{1}}_A(t)}^2 \\
&= \alpha^3 + \sum_{t \neq 0} \hat{\mathbb{1}}_{2 \cdot A}(t)\overline{\hat{\mathbb{1}}_A(t)}^2
\end{aligned}$$

But

$$
\begin{aligned}
\left|\sum_{t\neq 0}\hat{\mathbb{1}}_{2\cdot A}(t)\overline{\hat{\mathbb{1}}_{A}(t)}^{2}\right| &\leq \sup_{t\neq 0}\left|\hat{\mathbb{1}}_A(t)\right|\sum_{t\neq 0}\left|\hat{\mathbb{1}}_{2\cdot A}(t)\right|\left|\hat{\mathbb{1}}_A(t)\right| \\
&\leq \varepsilon\sum_{t\in\hat{\mathbb{F}}_p^n}\left|\hat{\mathbb{1}}_{2\cdot A}(t)\right|\left|\hat{\mathbb{1}}_A(t)\right| \\
&\leq \varepsilon\left(\sum_t\left|\hat{\mathbb{1}}_{2\cdot A}(t)\right|^2\sum_t\left|\hat{\mathbb{1}}_A(t)\right|^2\right)^{1/2} && \text{by Cauchy-Schwarz} \\
&= \varepsilon\left\|\hat{\mathbb{1}}_{2\cdot A}\right\|_2\left\|\hat{\mathbb{1}}_A\right\|_2 \\
&= \varepsilon\cdot\alpha^2\leq\varepsilon && \text{by Parseval.}
\end{aligned}
$$

□

Theorem 2.36 (Meshulam) Let $A \subseteq \mathbb{F}_p^n$ be a set containing no non-trivial 3-APs. Then $|A| = O(p^n/\log p^n)$, i.e. $\alpha = O(1/n)$.

Proof (Hints).

- Use similar proof as that of above lemma to show that $\left|T_3(\mathbb{1}_A,\mathbb{1}_A,\mathbb{1}_A) - \alpha^3\right| \leq \sup_{t\neq 0}\left|\hat{\mathbb{1}}_A(t)\right|\cdot\alpha$.
- Reason that provided $p^n \geq 2\alpha^{-2}$, we have $\sup_{t\neq 0}\left|\hat{\mathbb{1}}_A(t)\right| \geq \frac{\alpha^2}{2}$.
- Use this to iteratively generate A_1, V_1, A_2, V_2, ...
- Reason that each A_i contains no non-trivial 3 APs.
- Find an expression for maximum number of steps it takes for the density of the A_i to increase from $2^k\alpha$ to $2^{k+1}\alpha$ (in terms of k and α). Use this to deduce an upper bound for the maximum number steps it takes for the density to reach 1.
- Find lower bound for $\dim(V_m)$ where V_m is the final V_i in the sequence, use fact that iteration halted to deduce that $p^{\dim(V_m)} \leq 2\alpha^{-2}$.
- Reason that we can assume $\alpha \geq \sqrt{2}p^{-n/4}$, and conclude that $\alpha \leq 16n$.

□

Proof. By assumption, $T_3(\mathbb{1}_A,\mathbb{1}_A,\mathbb{1}_A) = |A|/(p^n)^2 = \alpha/p^n$ (there are $|A|$ trivial APs). By the proof of the above lemma,

$$\left|T_3(\mathbb{1}_A,\mathbb{1}_A,\mathbb{1}_A) - \alpha^3\right| \leq \sup_{t\neq 0}\left|\hat{\mathbb{1}}_A(t)\right|\cdot\alpha.$$

So provided that $p^n \geq 2\alpha^{-2}$, we have $T_3(\mathbb{1}_A,\mathbb{1}_A,\mathbb{1}_A) \leq \alpha^3/2$, so $\left|T_3(\mathbb{1}_A,\mathbb{1}_A,\mathbb{1}_A) - \alpha^3\right| \geq \alpha^3/2$, hence

$$\sup_{t\neq 0}\left|\hat{\mathbb{1}}_A(t)\right| \geq \frac{\alpha^2}{2}.$$

So by Lemma 2.32 with $\rho = \frac{\alpha}{2}$, there exists a subspace $V \leq \mathbb{F}_p^n$ of codimension 1 and $x \in \mathbb{F}_p^n$ such that $|A \cap (x+V)| \geq (\alpha + \alpha^2/4)|V|$.

We iterate this observation: let $A_0 = A$, $V_0 = \mathbb{F}_p^n$, $\alpha_0 = |A_0|/|V_0|$. At this i-th step, we are given a set $A_{i-1} \subseteq V_{i-1}$ of density α_{i-1} with no non-trivial 3-APs. Provided that

$p^{\dim(V_{i-1})} \geq 2\alpha_{i-1}^{-2}$, there exists a subspace $V_i \leq V_{i-1}$ of codimension 1 and $x_i \in V_{i-1}$ such that

$$|(A - x_i) \cap V_i| = |A \cap (x_i + V_i)| \geq (\alpha_{i-1} + \alpha_{i-1}^2/4)|V_i|$$

So set $A_i = (A - x_i) \cap V_i$. A_i has density $\alpha_i \geq \alpha_{i-1} + \alpha_{i-1}^2/4$, and contains no non-trivial 3-APs (since the translate $A - x_i$ contains no non-trivial 3-APs). Through this iteration, the density increases:

- from α to 2α in at most $\alpha/(\alpha^2/4) = 4\alpha^{-1}$ steps,
- from 2α to 4α in at most $(2\alpha)/((2\alpha)^2/4) = 2\alpha^{-1}$ steps.
- and so on, …

So the density reaches 1 in at most $4\alpha^{-1}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) = 8\alpha^{-1}$ steps. The iteration must end with $\dim(V_i) \geq n - 8\alpha^{-1}$, at which point we must have had $p^{\dim(V_i)} < 2\alpha_{i-1}^{-2} \leq 2\alpha^{-2}$, or else we could have iterated again.

But we may assume that $\alpha \geq \sqrt{2}p^{-n/4}$ (since otherwise we would be done), so $\alpha^{-2} < \frac{1}{2}p^{n/2}$, whence $p^{n-8\alpha^{-1}} \leq p^{n/2}$, i.e. $\frac{n}{2} \leq 8\alpha^{-1}$. □

Remark 2.37 The current largest known subset of $\mathbb{F}_3^n$ containing no non-trivial 3-APs has size 2.2202^n.

Lemma 2.38 Let $A \subseteq [N]$ be of density $\alpha > 0$ and contain no non-trivial 3-APs, with $N > 50\alpha^{-2}$. Let p be a prime with $p \in [N/3, 2N/3]$, and write $A' = A \cap [p] \subseteq \mathbb{Z}/p$. Then one of the following holds:

1. $\sup_{t \neq 0} \left|\widehat{\mathbb{1}}_{A'}(t)\right| \geq \alpha^2/10$ (where the Fourier coefficient is computed in $\mathbb{Z}/p$).
2. There exists an interval $J \subseteq [N]$ of length $\geq N/3$ such that $|A \cap J| \geq \alpha(1 + \alpha/400)|J|$.

Proof (Hints).

- Show that we can assume $|A'| \geq \alpha(1 - \alpha/200)p$.

□

Proof. TODO: fill in details in proof.

We may assume that $|A'| = |A \cap [p]| \geq \alpha(1 - \alpha/200)p$, since otherwise $|A \cap [p + 1, N]| \geq \alpha N - (\alpha(1 - \alpha/200)p) = \alpha(N - p) + \frac{\alpha^2}{200}p \geq (\alpha + \alpha^2/400)(N - p)$ since $p \geq N/3$, which implies case 2 with $J = [p + 1, N]$.

Let $A'' = A' \cap [p/3, 2p/3]$. Note that all 3-APs of the form $(x, x + d, x + 2d) \in A' \times A'' \times A''$ are in fact APs in $[N]$. If $|A' \cap [p/3]|$ or $|A' \cap [2p/3, p]|$ is at least $\frac{2}{5}|A'|$, then again we are in case 2. So we may assume that $|A''| \geq |A'|/5$. Now as in above lemmas, we have

$$\frac{\alpha''}{p} = \frac{|A''|}{p^2} = T_3(\mathbb{1}_{A'}, \mathbb{1}_{A''}, \mathbb{1}_{A''}) = \alpha'(\alpha'')^2 + \sum_t \widehat{\mathbb{1}}_{A'}(t)\widehat{\mathbb{1}}_{A''}(t)\overline{\widehat{\mathbb{1}}_{2\cdot A''}(t)}$$

where $\alpha' = |A'|/p$ and $\alpha'' = |A''|/p$. So as before,

$$\frac{\alpha'\alpha''}{2} \leq \sup_{t\neq 0}|\mathbb{1}_{A'}(t)| \cdot \alpha''$$

provided that $\alpha''/p \leq \frac{1}{2}\alpha'(\alpha'')^2$, i.e. $2/p \leq \alpha'\alpha''$ (check this inequality indeed holds). Hence, $\sup_{t\neq 0}\left|\hat{\mathbb{1}}_{A'}(t)\right| \geq \frac{\alpha'\alpha''}{2} \geq \frac{1}{2}\alpha(1-\alpha/200)^2 \cdot \frac{2}{5} \geq \alpha^2/10$. TODO: constants need to change somewhere here. □

Lemma 2.39 Let $m \in \mathbb{N}$, and let $\varphi : [m] \to \mathbb{Z}/p$ be given by $\varphi(x) = tx$ for some $t \neq 0$. For all $\varepsilon > 0$, there exists a partition of $[m]$ into progressions P_i of length $\ell_i \in [\varepsilon\sqrt{m}/2, \varepsilon\sqrt{m}]$, such that

$$\forall i, \quad \operatorname{diam}(\varphi(P_i)) := \max_{x,y\in P_i}|\varphi(x) - \varphi(y)| \leq \varepsilon p$$

(where $|\varphi(x) - \varphi(y)|$ views $\varphi(x), \varphi(y) \in \{0, ..., p-1\}$).

Proof. Let $u = \lfloor\sqrt{m}\rfloor$ and consider $0, t, ..., ut$. By the pigeonhole principle, there exists $0 \leq v < w \leq u$ such that $|wt - vt| = |(w-v)t| \leq p/u$. Set $s = w - v$, so $|st| \leq p/u$. Divide $[m]$ into residue classes $\bmod s$, each of which has size at least $m/s \geq m/u$. But each residue class can be divided into APsof the form $a, a+s, ..., a+ds$ for some $\varepsilon u/2 < d \leq \varepsilon u$. The diameter of the image of each progression under φ is $|dst| \leq dp/u \leq \varepsilon up/u = \varepsilon p$. □

Lemma 2.40 Let $A \subseteq [N]$ be of density $\alpha > 0$, let p be prime with $p \in [N/3, 2N/3]$, and write $A' = A \cap [p] \subseteq \mathbb{Z}/p$. Suppose that $\left|\hat{\mathbb{1}}_{A'}(t)\right| \geq \alpha^2/20$ for some $t \neq 0$. Then there exists a progression $P \subseteq [N]$ of length at least $\alpha^2\sqrt{N}/500$ such that $|A \cap P| \geq \alpha(1+\alpha/80)|P|$.

Proof. Let $\varepsilon = \alpha^2/40\pi$ and use above lemma to partition $[p]$ into progressions P_i of length $\geq \varepsilon\sqrt{p/2} \geq \alpha^2/40\pi\frac{\sqrt{N/3}}{2} \geq \alpha^{\sqrt{N}}/500$, and $\operatorname{diam}(\varphi(P_i)) \leq \varepsilon p$. Fix one x_i from each of the P_i. Then

$$\begin{aligned}
\frac{\alpha^2}{20} \leq \left|\hat{f}_{A'}(t)\right| &= \frac{1}{p}\sum_i\sum_{x\in P_i} f_{A'}(x)e(-xt/p) \\
&= \frac{1}{p}\left|\sum_i\sum_{x\in P_i} f_{A'}(x)e(-xit/p) + \sum_i\sum_{x\in P_i} f_{A'}(x)(e(-xt/p) - e(-xit/p))\right| \\
&\leq \frac{1}{p}\sum_i\left|\sum_{x\in P_i} f_{A'}(x)\right| + \frac{1}{p}\sum_i\sum_{x\in P_i}|f_{A'}(x)|\underbrace{|e(-xt/p) - e(-xit/p)|}_{\leq 2\pi\varepsilon \text{ since } \operatorname{diam}(\varphi(P_i))\leq\varepsilon p}
\end{aligned}$$

So

$$\sum_i\left|\sum_{x\in P_i} f_{A'}(x)\right| \geq \frac{\alpha^2}{40}p$$

Since $f_{A'}$ has mean zero,

$$\sum_i \left(\left| \sum_{x \in P_i} f_{A'}(x) \right| + \sum_{x \in P_i} f_{A'}(x) \right) \geq \frac{\alpha^2}{40} p$$

hence $\exists i$ such that

$$\left| \sum_{x \in P_i} f_{A'}(x) \right| + \sum_{x \in P_i} f_{A'}(x) \geq \frac{\alpha^2}{80} |P_i|$$

and so

$$\sum_{x \in P_i} f_{A'}(x) \geq \frac{\alpha^2}{160} |P_i|.$$

□

Definition 2.41 Let $\Gamma \subseteq \hat{G}$ and $\rho > 0$. The **Bohr set** $B(\Gamma, \rho)$ is the set

$$B(\Gamma, \rho) = \{x \in G : |\gamma(x) - 1|) < \rho \ \forall \gamma \in \Gamma\}.$$

The **rank** of $B(\Gamma, \rho)$ is $|B(\Gamma, \rho)|$, and is **width** (or **radius**) is ρ.

Example 2.42 Let $G = \mathbb{F}_p^n$, then $B(\Gamma, \rho) = \langle \Gamma \rangle^\perp$ for all sufficiently small ρ. Here, the rank gives an upper bound on $\mathrm{codim}(\langle \Gamma \rangle^\perp)$.

Lemma 2.43 Let $\Gamma \subseteq \hat{G}$ and $|\Gamma| = d$, and let $\rho > 0$. Then

$$|B(\Gamma, \rho)| \geq \left(\frac{\rho}{8} \right)^d |G|.$$

Proposition 2.44 (Bogolyubov's Lemma for Finite Abelian Groups) Let $A \subseteq G$ be of density $\alpha > 0$. Then there exists $\Gamma \subseteq \hat{G}$ with $|\Gamma| \leq 2\alpha^{-2}$ such that

$$B\left(\Gamma, \frac{1}{2} \right) \subseteq A + A - (A + A).$$

Proof. Recall that

$$(\mathbb{1}_A * \mathbb{1}_A * \mathbb{1}_A * \mathbb{1}_A)(x) = \sum_{\gamma \in \widehat{G}} \left| \hat{\mathbb{1}}_A(\gamma) \right|^4 \gamma(x)$$

Let $\Gamma = \mathrm{Spec}_{\sqrt{\alpha/2}}(\mathbb{1}_A)$ and note that for $x \in B(\Gamma, 1/2)$ and $\gamma \in \Gamma$, $\mathrm{Re}(\gamma(x)) > 0$. Hence, for $x \in B(\Gamma, 1/2)$,

$$\mathrm{Re}\left(\sum_{\gamma \in \widehat{G}} \left| \hat{\mathbb{1}}_A(\gamma) \right|^4 \gamma(x) \right) = \mathrm{Re}\left(\sum_{\gamma \in \Gamma} \right) \left| \hat{\mathbb{1}}_A(\gamma) \right|^4 \gamma(x)) + \mathrm{Re}\left(\sum_{x \notin \Gamma} \right) \left| \hat{\mathbb{1}}_A(\gamma) \right|^4 \gamma(x))$$

and

$$\left|\mathrm{Re}\left(\sum_{\gamma\notin\Gamma}\left|\hat{\mathbb{1}}_A(\gamma)\right|^4\gamma(x)\right)\right|) \le \sup_{\gamma\notin\Gamma}\left|\hat{\mathbb{1}}_A(\gamma)\right|^2\sum_{\gamma\notin\Gamma}\left|\hat{\mathbb{1}}_A(\gamma)\right|^2$$
$$\le \left(\sqrt{\frac{\alpha}{2}}\cdot\alpha\right)^2\cdot\alpha = \frac{\alpha^4}{2}$$

by Parseval. □

Theorem 2.45 (Roth) Let $A \subseteq [N]$ be a set containing no non-trivial 3-APs. Then $|A| = O(N/\log\log N)$.

Proof.

□

Example 2.46 (Behrend's Example) There exists a set $A \subseteq [N]$ of size $|A| \ge \exp(-c\sqrt{\log N})N$ containing no non-trivial 3-APs.

3. Probabilistic tools

All probability spaces here will be finite.

Theorem 3.1 (Khintchine's Inequality) Let $p \in [2, \infty)$. Let $X_1, ..., X_n$ be independent random variables such that

$$\forall i \in [n], \quad \mathbb{P}(X_i = x_i) = \mathbb{P}(X_i = -x_i) = \frac{1}{2}$$

for some $x_1, ..., x_n \in \mathbb{C}$. Then

$$\left\|\sum_{i=1}^n X_i\right\|_{L^p(\mathbb{P})} = O\left(p^{1/2}\left(\sum_{i=1}^n \|X_i\|^2_{L^2(\mathbb{P})}\right)^{1/2}\right)$$

Proof (Hints).

- Explain why sufficient to prove for the case that $p = 2k$ for $k \in \mathbb{N}$.
- Explain why $\sum_{i=1}^n \|X_i\|^2_{L^\infty(\mathrm{Pr})} = \sum_{i=1}^n \|X_i\|^2_{L^2(\mathrm{Pr})}$, and assume they are equal to 1.
- Show that $\|X\|^{2k}_{L^{2k}(\mathrm{Pr})} \le 8kI(k)$, where $I(k) = \int_0^\infty t^{2k-1}\exp(-t^2/4)\,\mathrm{d}t$.
- Show by induction on k that $I(k) \le 2^{2k}(2k)^k/4k$.

□

Proof. Since L^p norms are nested, it suffices to prove in the case that $p = 2k$ for some $k \in \mathbb{N}$. Write $X = \sum_{i=1}^n X_i$, and assume the quantity $\sum_{i=1}^n \|X_i\|^2_{L^\infty(\mathbb{P})} = \sum_{i=1}^n |x_i|^2 = \sum_{i=1}^n \|X_i\|^2_{L^2(\mathbb{P})}$ is equal to 1. By Chernoff's Inequality, $\forall\theta > 0$,

$$\Pr(|X| \ge \theta) \le 4\exp(-\theta^2/4),$$

and so, since $\int_0^t P_X(s)\,\mathrm{d}s = \Pr(|X| \le t)$,

$$
\begin{aligned}
\|X\|_{L^{2k}(\Pr)}^{2k} &= \int_0^\infty t^{2k} P_X(t)\,\mathrm{d}t \\
&= \int_0^\infty 2kt^{2k-1}\Pr(|X|\geq t)\,\mathrm{d}t \text{ by integration by parts} \\
&\leq 8k\int_0^\infty t^{2k-1}\exp(-t^2/4)\,\mathrm{d}t =: 8kI(k)
\end{aligned}
$$

We will show by induction on k that $I(k) \leq 2^{2k}(2k)^k/4k$. Indeed, when $k=1$,

$$
\begin{aligned}
\int_0^\infty t\exp(-t^2/4)\,\mathrm{d}t &= \left[-2\exp(-t^2/4)\right]_0^\infty = 2 \\
&= 2^{2\cdot 1}(2\cdot 1)^1/(4\cdot 1)
\end{aligned}
$$

For $k>1$, we integrate by parts to find that

$$
\begin{aligned}
I(k) &:= \int_0^\infty \underbrace{t^{2k-2}}_{u} \cdot \underbrace{t\exp(-t^2/4)}_{v'}\,\mathrm{d}t \\
&= \left[t^{2k-2}\cdot(-2\exp(-t^2/4))\right]_0^\infty - \int_0^\infty (2k-2)t^{2k-3}\cdot(-2\exp(-t^2/4))\,\mathrm{d}t \\
&= 4(k-1)\int_0^\infty t^{2(k-1)-1}\exp(-t^2/4)\,\mathrm{d}t \\
&= 4(k-1)I(k-1) \\
&\leq \frac{4(k-1)2^{2k-1}(2(k-1))^{k-1}}{4(k-1)} \text{ by induction hypothesis} \\
&\leq \frac{2^{2k}(2k)^k}{4k}.
\end{aligned}
$$

□

Corollary 3.2 (Rudin's Inequality) Let $\Gamma \subseteq \widehat{\mathbb{F}}_2^n$ be a linearly independent set and let $p\in[2,\infty)$. Then $\forall \hat{f}\in\ell^2(\Gamma)$,

$$
\left\|\sum_{\gamma\in\Gamma}\hat{f}(\gamma)\gamma\right\|_{L^p(\mathbb{F}_2^n)} = O\left(\sqrt{p}\cdot\left\|\hat{f}\right\|_{\ell^2(\Gamma)}\right)
$$

Proof. Exercise. □

Corollary 3.3 (Dual Rudin) Let $\Gamma \subseteq \widehat{\mathbb{F}}_2^n$ be a linearly independent set and let $p\in(1,2]$. Then $\forall f\in L^p(\mathbb{F}_2^n)$,

$$
\left\|\hat{f}\right\|_{\ell^2(\Gamma)} = O\left(\sqrt{\frac{p}{p-1}}\cdot\|f\|_{L^p(\mathbb{F}_2^n)}\right).
$$

Proof (Hints). Let $g(x)=\sum_{\gamma\in\Gamma}\hat{f}(\gamma)\gamma(x)$. Show that $\left\|\hat{f}\right\|_{\ell^2(\Gamma)}^2 \leq \|f\|_{L^p(\mathbb{F}_2^n)}\|g\|_{L^q(\mathbb{F}_2^n)}$ where $1/p+1/q=1$, and conclude using Rudin's Inequality. □

Proof. Let $f \in L^p(\mathbb{F}_2^n)$ and let $g(x) = \sum_{\gamma\in\Gamma} \hat{f}(\gamma)\gamma(x)$. Then

$$\begin{aligned}
\left\|\hat{f}\right\|^2_{\ell^2(\Gamma)} &:= \sum_{\gamma\in\Gamma} \left|\hat{f}(\gamma)\right|^2 \\
&= \langle \hat{f}, \hat{g}\rangle_{\ell^2(\Gamma)} = \langle \hat{f}, \hat{g}\rangle_{\ell^2\left(\widehat{\mathbb{F}_2^n}\right)} \\
&= \langle f, g\rangle_{L^2(\mathbb{F}_2^n)} && \text{by Plancherel's Identity} \\
&\le \|f\|_{L^p(\mathbb{F}_2^n)}\|g\|_{L^q(\mathbb{F}_2^n)} && \text{by Hölder's Inequality.}
\end{aligned}$$

where $1/p + 1/q = 1$. By Rudin's Inequality,

$$\begin{aligned}
\|g\|_{L^q(\mathbb{F}_2^n)} &= O\left(\sqrt{q}\cdot\|\hat{g}\|_{\ell^2(\Gamma)}\right) \\
&= O\left(\sqrt{\frac{p}{p-1}}\cdot\left\|\hat{f}\right\|_{\ell^2(\Gamma)}\right).
\end{aligned}$$

□

Recall that given $A \subseteq \mathbb{F}_2^n$ of density $\alpha > 0$, we have $\left|\mathrm{Spec}_\rho(\mathbb{1}_A)\right| \le \rho^{-2}\alpha^{-1}$. This is the best possible bound as the example of a subspace A shows. However, in this case, the large spectrum is highly structured.

Theorem 3.4 (Special Case of Chang's Theorem) Let $A \subseteq \mathbb{F}_2^n$ be of density $\alpha > 0$. Then

$$\forall\rho > 0, \exists H \le \widehat{\mathbb{F}_2^n} : \dim(H) = O(\rho^{-2}\log\alpha^{-1}) \quad \text{and} \quad \mathrm{Spec}_\rho(\mathbb{1}_A) \subseteq H.$$

Proof (Hints). Use Dual Rudin on a maximal linearly independent set in $\mathrm{Spec}_\rho(\mathbb{1}_A)$, with $p = 1 + (\log\alpha^{-1})^{-1}$. □

Proof. Let $\Gamma \subseteq \mathrm{Spec}_\rho(\mathbb{1}_A)$ be maximal linearly independent set. Let $H = \langle\mathrm{Spec}_\rho(\mathbb{1}_A)\rangle$. Clearly $\dim(H) = |\Gamma|$. By Dual Rudin, $\forall p \in (1, 2]$,

$$(\rho\alpha)^2|\Gamma| \le \sum_{\gamma\in\Gamma}\left|\hat{\mathbb{1}}_A(\gamma)\right|^2 = \left\|\hat{\mathbb{1}}_A\right\|^2_{\ell^2(\Gamma)} = O\left(\frac{p}{p-1}\|\mathbb{1}_A\|^2_{L^p(\mathbb{F}_2^n)}\right) = O\left(\frac{p}{p-1}\alpha^{2/p}\right).$$

Hence, $|\Gamma| \le O\left(\rho^{-2}\alpha^{-2}\alpha^{2/p}\frac{p}{p-1}\right)$. Setting $p = 1 + (\log\alpha^{-1})^{-1}$, we obtain $|\Gamma| \le O(\rho^{-2}\alpha^{-2}(\alpha^2 e^2)(\log\alpha^{-1} + 1)) = O(\rho^{-2}\log\alpha^{-1})$. □

Definition 3.5 Let G be a finite abelian group. $S \subseteq G$ is **dissociated** if, whenever $\sum_{s\in S}\varepsilon_s s = 0$ with each $\varepsilon_s \in \{-1, 0, 1\}$, then we have $\varepsilon_s = 0$ for all $s \in S$.

Example 3.6 Clearly, if $G = \mathbb{F}_2^n$, then $S \subseteq G$ is dissociated iff S is linearly independent.

Theorem 3.7 (Chang) Let G be a finite abelian group, and let $A \subseteq G$ be of density $\alpha > 0$. If $\Lambda \subseteq \mathrm{Spec}_\rho(\mathbb{1}_A)$ is dissociated, then $|\Lambda| = O(\rho^{-2}\log\alpha^{-1})$.

Theorem 3.8 (Marcinkiewicz-Zygmund) Let $p \in [2, \infty)$ and let $X_1, \ldots, X_n \in L^p(\mathrm{Pr})$ be independent RVs with $\mathbb{E}[X_1 + \cdots + X_n] = 0$. Then

$$\left\|\sum_{i=1}^{n} X_i\right\|_{L^p(\Pr)} = O\left(p^{1/2} \cdot \left\|\sum_{i=1}^{n} |X_i|^2\right\|_{L^{p/2}(\Pr)}^{1/2}\right).$$

Proof. First assume that the distribution of the X_i is symmetric, i.e. $\Pr(X_i = a) = \Pr(X_i = -a)$ for all $a \in \mathbb{R}$ and $i \in [n]$. Partition the probability space Ω into sets $\Omega_1, \Omega_2, ..., \Omega_M$ and write $\Pr_j$ for the induced measure on Ω, such that all X_i are symmetric and take at most 2 values. By Khintchine's inequality, for each $j \in [M]$,

$$\begin{aligned}\left\|\sum_{i=1}^{n} X_i\right\|_{L^p(\Pr_j)}^p &= O\left(p^{p/2} \cdot \left(\sum_{i=1}^{n} \|X_i\|_{L^2(\Pr_j)}^2\right)^{p/2}\right)\\ &= O\left(p^{p/2} \cdot \left\|\sum_{i=1}^{n} |X_i|^2\right\|_{L^{p/2}(\Pr_j)}^{p/2}\right).\end{aligned}$$

Summing over all $j \in [M]$ and taking p-th roots gives the result for the symmetric case.

Now suppose the X_i are arbitrary RVs, and let $Y_1, ..., Y_n$ be such that $Y_i \sim X_i$ and $X_1, Y_1, ..., X_n, Y_n$ are all independent. Applying the symmetric case to the RVs $X_i - Y_i$, we obtain

$$\begin{aligned}\left\|\sum_{i=1}^{n} (X_i - Y_i)\right\|_{L^p(\Pr \times \Pr)} &= O\left(p^{1/2} \cdot \left\|\sum_{i=1}^{n} |X_i - Y_i|^2\right\|_{L^{p/2}(\Pr \times \Pr)}^{1/2}\right)\\ &= O\left(p^{1/2} \cdot \left\|\sum_{i=1}^{n} |X_i^2|\right\|_{L^{p/2}(\Pr)}^{1/2}\right) \quad \text{TODO: check this explicitly}\end{aligned}$$

But then

$$\begin{aligned}\left\|\sum_{i=1}^{n} X_i\right\|_{L^p(\Pr)}^p &= \left\|\sum_{i=1}^{n} X_i - \mathbb{E}_Y\left[\sum_{i=1}^{n} Y_i\right]\right\|_{L^p(\Pr)}^p\\ &= \mathbb{E}_X\left|\sum_{i=1}^{n} X_i - \mathbb{E}_Y\left[\sum_{i=1}^{n} Y_i\right]\right|^p\\ &= \mathbb{E}_X\left|\mathbb{E}_Y \sum_{i=1}^{n} (X_i - Y_i)\right|^p\\ &\leq \mathbb{E}_X\mathbb{E}_Y\left|\sum_{i=1}^{n} (X_i - Y_i)\right|^p \quad \text{by Jensen's inequality}\\ &= \left\|\sum_{i=1}^{n} (X_i - Y_i)\right\|_{L^p(\Pr \times \Pr)}^p.\end{aligned}$$

□

Theorem 3.9 (Croot-Sisask Almost Periodicity) Let G be a finite abelian group, let $\varepsilon > 0$, and $p \in [2, \infty)$. Let $A, B \subseteq G$ be such that $|A + B| \leq K|A|$, and let $f : G \to \mathbb{C}$. Then there is $b \in B$ and a set $X \subseteq B - b$ such that $|X| \geq \frac{1}{2}K^{-O(\varepsilon^{-2}p)}|B|$ and

$$\|\tau_x(f * \mu_A) - f * \mu_A\|_{L^p(G)} \leq \varepsilon \|f\|_{L^p(G)} \quad \forall x \in X,$$

where $\tau_x g(y) = g(y + x)$ for all $y \in G$.

Proof. The main idea is to approximated

$$(f * \mu_A)(y) = \mathbb{E}_{x \in G} f(y - x)\mu_A(x) = \mathbb{E}_{x \in A} f(y - x)$$

by $\frac{1}{m}\sum_{i=1}^m f(y - z_i)$ where the z_i are sampled independently and uniformly from A, and m is to be chosen later. For each $y \in G$, define $Z_i(y) = \tau_{-z_i} f(y) - (f * \mu_A)(y)$. For each $y \in G$, these are independent random variables with mean 0. So by Marcinkiewicz-Zygmund,

$$\begin{aligned}
\left\|\sum_{i=1}^m Z_i(y)\right\|_{L^p(\mathrm{Pr})}^p &= O\left(p^{p/2} \cdot \left\|\sum_{i=1}^m |Z_i(y)|^2\right\|_{L^{p/2}(\mathrm{Pr})}^{p/2}\right) \\
&= O\left(p^{p/2} \cdot \mathbb{E}_{(z_1,\dots,z_m)\in A^m}\left|\sum_{i=1}^m |Z_i(y)|^2\right|^{p/2}\right).
\end{aligned}$$

By Holder's inequality with $1/p' + 2/p = 1$,

$$\begin{aligned}
\left|\sum_{i=1}^m |Z_i(y)|^2\right|^{p/2} &\leq \left(\sum_{i=1}^m 1^{p'}\right)^{\frac{1}{p'}\cdot\frac{p}{2}} \cdot \left(\sum_{i=1}^m |Z_i(y)|^{2\cdot\frac{p}{2}}\right)^{\frac{2}{p}\cdot\frac{p}{2}} \\
&= m^{p/2-1} \cdot \sum_{i=1}^m |Z_i(y)|^p.
\end{aligned}$$

So

$$\left\|\sum_{i=1}^m Z_i(y)\right\|_{L^p(\mathrm{Pr})}^p = O\left(p^{p/2}m^{p/2-1} \cdot \mathbb{E}_{(z_1,\dots,z_m)\in A^m}\sum_{i=1}^m |Z_i(y)|^p\right).$$

Summing over all $y \in G$, we have

$$\mathbb{E}_{y\in G}\left\|\sum_{i=1}^m Z_i(y)\right\|_{L^p(\mathrm{Pr})}^p = O\left(p^{p/2}m^{p/2-1}\mathbb{E}_{(z_1,\dots,z_m)\in A^m}\sum_{i=1}^m \mathbb{E}_{y\in G}|Z_i(y)|^p\right)$$

and $\left(\mathbb{E}_{y\in G}|Z_i(y)|^p\right)^{1/p} = \|Z_i\|_{L^p(G)} = \left\|\tau_{-z_i} f - f * \mu_A\right\|_{L^p(G)} \leq \left\|\tau_{-z_i} f\right\|_{L^p(G)} + \|f * \mu_A\|_{L^p(G)} \leq \|f\|_{L^p(G)} + \|f\|_{L^p(G)} \cdot \|\mu_A\|_{L^1(G)} \leq 2\|f\|_{L^p(G)}$ by Young's convolution inequality. So we have

$$\mathbb{E}_{(z_1,\dots,z_m)\in A^m}\mathbb{E}_{y\in G}\left|\sum_{i=1}^m Z_i(y)\right|^p = O\left(p^{p/2}m^{p/2-1}\sum_{i=1}^m \left(2\|f\|_{L^p(G)}\right)^p\right)$$
$$= O\big((4p)^{p/2}m^{p/2}\|f\|_{L^p(G)}^p\big).$$

Choose $m = O(\varepsilon^{-2}p)$ so that the RHS is at most $\left(\frac{\varepsilon}{4}\|f\|_{L^p(G)}\right)^p$, and f§or $(z_1,\dots,z_m)\in A^m$, define

$$M_{(z_1,\dots,z_m)} := \mathbb{E}_{y\in G}\left|\frac{1}{m}\sum_{i=1}^m \tau_{-z_i}f(y) - (f*\mu_A)(y)\right|^p.$$

Then we have

$$\mathbb{E}_{(z_1,\dots,z_m)\in A^m}M_{(z_1,\dots,z_m)} = O\big((4p)^{p/2}m^{p/2}\|f\|_{L^p(G)}^p\big) = \left(\frac{\varepsilon}{4}\|f\|_{L^p(G)}\right)^p.$$

Also define

$$L = \left\{\boldsymbol{z}\in A^m : M_{\boldsymbol{z}} \le \left(\frac{\varepsilon}{2}\|f\|_{L^p(G)}\right)^p\right\}.$$

By Markov's inequality, since

$$\mathbb{E}_{\boldsymbol{z}\in A^m}M_{\boldsymbol{z}} \le \left(\frac{\varepsilon}{4}\|f\|_{L^p(G)}\right)^p = 2^{-p}\left(\frac{\varepsilon}{2}\|f\|_{L^p(G)}\right)^p,$$

we have

$$\frac{|A^m\setminus L|}{|A^m|} = \Pr\left(M_{\boldsymbol{z}} \ge \left(\frac{\varepsilon}{2}\|f\|_{L^p(G)}\right)^p\right) \le \Pr(M_{\boldsymbol{z}} \ge 2^p\mathbb{E}_{\boldsymbol{z}\in A^m}M_{\boldsymbol{z}}) \le 2^{-p},$$

hence $|L| \ge (1-1/2^p)|A|^m \ge \frac{1}{2}|A|^m$. Let $D = \{(b,\dots,b) : b\in B\}\subseteq B^m$. Then $L + D\subseteq (A+B)^m$, and so

$$|L+D| \le |A+B|^m \le K^m|A|^m \le 2K^m|L|.$$

By Lemma 1.24,

$$E(L,D) \ge \frac{|L|^2|D|^2}{|L+D|} \ge \frac{1}{2}K^{-m}|D|^2|L|,$$

so there are at least $|D|^2/2K^m$ pairs $(d_1,d_2)\in D^2$ such that $r_{L-L}(d_2-d_1) > 0$. In particular, there exists $b\in B$ and $X\subseteq B-b$ such that $|X| \ge |D|/2K^m = |B|/2K^m$ and for all $x\in X$, there exists $\ell_2(x)\in L$ such that for all $\in [m]$, $\ell_1(x)_i - \ell_2(x)_i = x$. But now for all $x\in X$, by the triangle inequality, we have,

$$
\begin{aligned}
\|\tau_{-x}f * \mu_A - f * \mu_A\|_{L^p(G)} &\le \left\| \tau_{-x}f * \mu_A - \tau_{-x}\left(\frac{1}{m}\sum_{i=1}^{m} \tau_{-\ell_2(x)_i} f\right) \right\|_{L^p(G)} \\
&\quad + \left\| \tau_{-x}\left(\frac{1}{m}\sum_{i=1}^{m} \tau_{-\ell_2(x)_i f} - f * \mu_A\right) \right\|_{L^p(G)} \\
&= \left\| f * \mu_A - \frac{1}{m}\sum_{i=1}^{m} \tau_{-\ell_2(x)_i f} \right\|_{L^p(G)} \\
&\quad + \left\| \frac{1}{m}\sum_{i=1}^{m} \tau_{-x-\ell_2(x)_i f} - f * \mu_A \right\|_{L^p(G)} \\
&\le 2 \cdot \frac{\varepsilon}{2} \|f\|_{L^p(G)}
\end{aligned}
$$

by definition of L. □

Theorem 3.10 (Bogolyubov, after Sanders) Let $A \subseteq \mathbb{F}_p^n$ have density $\alpha > 0$. There exists a subspace $V \le \mathbb{F}_p^n$ of codimension $O\left(\left(\log \alpha^{-1}\right)^4\right)$ such that

$$V \subseteq (A + A) - (A + A)$$

4. Further topics

Theorem 4.1 (Ellenberg-Gijswijt) If $A \subseteq \mathbb{F}_3^n$ contains no non-trivial 3-term APs, then $|A| = o(2.756^n)$.

Notation 4.2 Let M_n denote the set of monomials in $x_1, ..., x_n$ whose degree in each variable is at most 2.

Notation 4.3 Let V_n denote the vector space of polynomials over $\mathbb{F}_3$ whose basis is M_n.

Notation 4.4 For any $0 \le d \le 2n$, let M_n^d denote the set of monomials in M_n of total degree at most d, and let V_n^d denote the corresponding vector space of polynomials. Write $m_d = \dim\left(V_n^d\right) = \left|M_n^d\right|$.

Lemma 4.5 Let $A \subseteq \mathbb{F}_3^n$ and $P \in V_n^d$ be a polynomial. If $P(a + b) = 0$ for all $a \ne b \in A$, then

$$|\{a \in A : P(2a) \ne 0\}| \le 2m_{d/2}.$$

Proof. Every $P \in V_n^d$ can be written as a linear combination of monomials in M_n^d, so

$$P(x + y) = \sum_{\substack{m,m' \in M_n^d \\ \deg(mm') \le d}} c_{m,m'} m(x) m'(y)$$

for some coefficients $c_{m,m'}$. Clearly, at least one of m, m' must have degree $\le d/2$, whence

$$P(x+y) = \sum_{m \in M_n^{d/2}} m(x) F_m(y) + \sum_{m' \in M_n^{d/2}} m'(y) G_{m'}(x),$$

for some families of polynomials $\left\{F_m : m \in M_n^{d/2}\right\}$ and $\left\{G_{m'} : m' \in M_n^{d/2}\right\}$. Viewing $(P(x+y))_{x,y \in A}$ as an $|A| \times |A|$ matrix C, we see that C can be written as the sum of at most $2m_{d/2}$ matrices, each of which has rank 1. Thus, $\operatorname{rank}(C) \leq 2m_{d/2}$. But by assumption, C is diagonal, and so its rank is equal to $|\{a \in A : P(a+a) \neq 0\}|$. □

Proposition 4.6 Let $A \subseteq \mathbb{F}_3^n$ be a set containing no non-trivial 3-APs. Then $|A| \leq 3m_{2n/3}$.

Proof. Let $d \in [0, 2n]$ be an integer which we will determine later. Let W be the space of polynomials in V_n^d that vanish in $(2 \cdot A)^c$. We have $\dim(W) \geq \dim\left(V_n^d\right) - |(2 \cdot A)^c| = m_d - (3^n - |A|)$.

We claim that there exists $P \in W$ such that $|\operatorname{supp}(P)| \geq \dim(W)$. Indeed, pick $P \in W$ with maximal support. If $|\operatorname{supp}(P)| < \dim(W)$, then there would be a non-zero polynomial $Q \in W$ vanishing on $\operatorname{supp}(P)$, in which case $\operatorname{supp}(P+Q) \supsetneq \operatorname{supp}(P)$, contradicting the maximality of $\operatorname{supp}(P)$.

Now by assumption, $\{a + a' : a \neq a' \in A\} \cap 2 \cdot A = \emptyset$, so any polynomial that vanishes on $(2 \cdot A)^c$ also vanishes on $\{a + a' : a \neq a' \in A\}$. Thus by above lemma,

$$\begin{aligned} m_d - (3^n - |A|) \leq \dim(W) \leq |\operatorname{supp}(P)| &= |\{x \in \mathbb{F}_3^n : P(x) \neq 0\}| \\ &= |\{a \in A : P(2a) \neq 0\}| \leq 2m_{d/2}. \end{aligned}$$

Hence, $|A| \leq 3^n - m_d + 2m_{d/2}$. But the monomials in $M_n \setminus M_n^d$ are in bijection with the ones in M_{2n-d} by $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \longleftrightarrow x_1^{2-\alpha_1} \cdots x_n^{2-\alpha_n}$, whence $3^n - m_d = m_{2n-d}$. Thus, setting $d = 4n/3$, we have

$$|A| \leq m_{2n/3} + 2m_{2n/3} = 3m_{2n/3}.$$

□

Example 4.7 Recall from (find lemma) that given $A \subseteq G$,

$$\left|T_3(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) - \alpha^3\right| \leq \sup_{\gamma \neq 1} \left|\widehat{\mathbb{1}}_A(\gamma)\right|.$$

However, it is impossible to bound $T_4(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) - \alpha^4$, where

$$T_4(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) = \mathbb{E}_{x,d} \mathbb{1}_A(x) \mathbb{1}_A(x+d) \mathbb{1}_A(x+2d) \mathbb{1}_A(x+3d),$$

by $\sup_{\gamma \neq 1} \left|\widehat{\mathbb{1}}_A(\gamma)\right|$. Indeed, consider $Q = \left\{x \in \mathbb{F}_p^n : x \cdot x = 0\right\}$. By (find example), $|Q|/p^n = 1/p + O\left(p^{-n/2}\right)$ and $\sup_{t \neq 0} \left|\widehat{\mathbb{1}}_Q(t)\right| = O\left(p^{-n/2}\right)$. But given a 3-AP $x, x+d, x+2d \in Q$, by the identity

$$\forall x, d, \quad x^2 - 3(x+d)^2 + 3(x+2d)^2 - (x+3d)^2 = 0,$$

$x + 3d$ automatically lies in Q, so $T_4(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) = T_3(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) = (1/p)^3 + O\left(p^{-n/2}\right)$.

Definition 4.8 Given $f : G \to \mathbb{C}$, define its $\boldsymbol{U^2}$**-norm** by

$$\|f\|_{U^2(G)}^4 = \mathbb{E}_{x,a,b\in G} f(x)\overline{f(x+a)f(x+b)}f(x+a+b)$$

By (find example), we have $\|f\|_{U^2(G)} = \left\|\hat{f}\right\|_{\ell^4(\widehat{G})}$, so it is indeed a norm.

Lemma 4.9 Let $f_1, f_2, f_3 : G \to \mathbb{C}$. Then

$$|T_3(f_1, f_2, f_3)| \le \min_{i\in[3]} \left(\|f_i\|_{U^2(G)} \cdot \prod_{j\ne i} \|f_j\|_{L^\infty(G)} \right).$$

Note that

$$\sup_{\gamma\in\widehat{G}} \left|\hat{f}(\gamma)\right|^4 \le \sum_{\gamma\in\widehat{G}} \left|\hat{f}(\gamma)\right|^4 \le \sup_{\gamma\in\widehat{G}} \left|\hat{f}(\gamma)\right|^2 \sum_{\gamma\in\widehat{G}} \left|\hat{f}(\gamma)\right|^2$$

and so by Parseval,

$$\left\|\hat{f}\right\|_{\ell^\infty(\widehat{G})} = \|f\|_{U^2(G)}^4 = \left\|\hat{f}\right\|_{\ell^\infty(\widehat{G})}^4 \le \left\|\hat{f}\right\|_{\ell^\infty(\widehat{G})}^2 \cdot \|f\|_{L^2(G)}^2.$$

Moreover, if $f = f_A = \mathbb{1}_A - \alpha$, then

$$T_3(f, f, f) = T_3(\mathbb{1}_A - \alpha, \mathbb{1}_A - \alpha, \mathbb{1}_A - \alpha) = T_3(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) - \alpha^3.$$

We may therefore reformulate the first step in the proof of Meshulam as follows: if $p^n \ge 2\alpha^{-2}$, then by (find lemma), $\frac{\alpha^3}{2} \le \left|\frac{\alpha}{p^n} - \alpha^3\right| = |T_3(f_A, f_A, f_A)| \le \|f_A\|_{U^2(\mathbb{F}_p^n)}$. It remains to show that if $\|f_a\|_{U^2(\mathbb{F}_p^n)}$ is non-trivial, then there exists a subspace $V \le \mathbb{F}_p^n$ of bounded codimension on which A has increased density.

Theorem 4.10 (U^2 Inverse Theorem) Let $f : \mathbb{F}_p^n \to \mathbb{C}$ satisfy $\|f\|_{L^\infty(\mathbb{F}_p^n)} \le 1$ and $\|f\|_{U^2(\mathbb{F}_p^n)} \ge \delta$ for some $\delta > 0$. Then there exists $b \in \mathbb{F}_p^n$ such that

$$\left|\mathbb{E}_{x\in\mathbb{F}_p^n} f(x)e(-x.b/p)\right| \ge \delta^2.$$

In other words, $\langle f, \varphi\rangle \ge \delta^2$ for $\varphi(x) = e(-x.b/p)$. We say "f correlates with a linear phase function".

Proof. We have seen that $\|f\|_{U^2(\mathbb{F}_p^n)} \le \left\|\hat{f}\right\|_{\ell^\infty(\mathbb{F}_p^n)} \|f\|_{L^2(\mathbb{F}_p^n)} \le \left\|\hat{f}\right\|_{\ell^\infty(\hat{\mathbb{F}}_p^n)}$. So

$$\delta^2 \le \left\|\hat{f}\right\|_{\ell^\infty(\hat{\mathbb{F}}_p^n)} = \sup_{t\in\hat{\mathbb{F}}_p^n} \left|\mathbb{E}_{x\in\mathbb{F}_p^n} f(x)e(-x.t/p)\right|.$$

□

Definition 4.11 Given $f : G \to \mathbb{C}$, the $\boldsymbol{U^3}$ **norm** of f is defined by

$$
\begin{aligned}
\|f\|_{U^3(G)}^8 &= \mathbb{E}_{x,a,b,c\in G} f(x)\overline{f(x+a)f(x+b)f(x+c)} \\
&\qquad\qquad f(x+a+b)f(x+b+c)f(x+a+c)\overline{f(x+a+b+c)} \\
&= \mathbb{E}_{x,h_1,h_2,h_3\in G} \prod_{\varepsilon\in\{0,1\}^3} \mathcal{C}^{|\boldsymbol{\varepsilon}|} f(x+\boldsymbol{\varepsilon}.\boldsymbol{h}),
\end{aligned}
$$

where $\mathcal{C}g(x) = \overline{g(x)}$ and $|\boldsymbol{\varepsilon}| = |\{i \in [3] : \varepsilon_i = 1\}|$ is the number of 1′s in $\boldsymbol{\varepsilon}$.

TODO: insert diagram of cube with vertices $x, x+a, ..., x+a+b+c$.

Remark 4.12 It is easy to verify that $\|f\|_{U^3(G)}^8 = \mathbb{E}_{c\in G}\|\Delta_c f\|_{U^2(G)}^4$ where $\Delta_c g(x) = g(x)\overline{g(x+c)}$.

Definition 4.13 Given eight functions $f_\varepsilon : G \to \mathbb{C}$ for $\varepsilon \in \{0,1\}^3$, define their ***U*³ inner product** by

$$
\left\langle (f_\varepsilon)_{\varepsilon\in\{0,1\}^3} \right\rangle_{U^3(G)} := \mathbb{E}_{x,h_1,h_2,h_3\in G} \prod_{\varepsilon\in\{0,1\}^3} \mathcal{C}^{|\varepsilon|} f_\varepsilon(x+\varepsilon\cdot h)
$$

Observe that $\langle f,f,f,f,f,f,f,f\rangle_{U^3(G)} = \|f\|_{U^3(G)}^8$.

Lemma 4.14 (Gowers-Cauchy-Schwarz Inequality) Given $f_\varepsilon : G \to \mathbb{C}$ for $\varepsilon \in \{0,1\}^3$,

$$
\left| \left\langle (f_\varepsilon)_{\varepsilon\in\{0,1\}^3} \right\rangle_{U^3(G)} \right| \le \prod_{\varepsilon\in\{0,1\}^3} \|f_\varepsilon\|_{U^3(G)}.
$$

Proof. Exercise (helpful to do for U^2 first). □

Remark 4.15 Setting $f_\varepsilon = f$ for $\varepsilon \in \{0,1\}^2 \times \{0\}$ and $f_\varepsilon = 1$ otherwise, it follows that

$$
\|f\|_{U^2(G)}^4 \le \|f\|_{U^3(G)}^4 \quad \text{hence} \quad \|f\|_{U^2(G)} \le \|f\|_{U^3(G)}.
$$

Proposition 4.16 Let $f_1, f_2, f_3, f_4 : \mathbb{F}_5^n \to \mathbb{C}$. Then

$$
|T_4(f_1,f_2,f_3,f_4)| \le \min_{i\in[4]} \|f_i\|_{U^3(G)} \cdot \prod_{j\neq i} \|f_j\|_{L^\infty(\mathbb{F}_5^n)}.
$$

Proof. Assume $f_i = f$ for all i and that $\|f\|_{L^\infty(\mathbb{F}_5^n)} \le 1$ (we can remove these assumptions). Reparameterising (by subtracting $a+b+c+d$), we have

$$
T_4(f,f,f,f) = \mathbb{E}_{a,b,c,d\in\mathbb{F}_5^n} f(3a+2b+c)f(2a+b-d)f(a-c-2d)f(-b-2c-3d)
$$

Now

$$
\begin{aligned}
|T_4(f,f,f,f)|^8 &\le \left(\mathbb{E}_{a,b,c}|\mathbb{E}_d f(2a+b-d)f(a-c-2d)f(-b-2c-3d)|^2\right)^4 \text{ by C-S} \\
&= \begin{pmatrix} \mathbb{E}_{d,d'}\mathbb{E}_{a,b} f(2a+b-d)\overline{f(2a+b-d')} \\ \cdot\, \mathbb{E}_c f(a-c-2d)\overline{f(a-c-2d')}f(-b-2c-3d)\overline{f(-b-2c-3d')} \end{pmatrix}^4 \\
&\le \mathbb{E}_{d,d'}\mathbb{E}_{a,b}\left|\mathbb{E}_c f(a-c-2d)\overline{f-c-2d'}f(-b-2c-3d)\overline{f(-b-2c-3d')}^2\right|^2 \\
&= \begin{pmatrix} \mathbb{E}_{c,c',d,d'}\mathbb{E}_a f(a-c-2d)\overline{f(a-c'-2d)f(a-c-2d')}f(a-c'-2d') \\ \cdot\, \mathbb{E}_b f(-b-2c-3d)\overline{f(-b-2c'-3d)f(-b-2c-3d')}f(-b-2c'-3d') \end{pmatrix}^2 \\
&\le \mathbb{E}_{c,c',d,d',a}\left|\mathbb{E}_b f(-b-2c-3d)\overline{f(-b-2c'-3d)f(-b-2c-3d')}f(-b-2c'-3d')\right|^2 \\
&= \mathbb{E}_{b,b',c,c',d,d'} f(-b-2c-3d)\overline{f(-b'-2c-3d)f(-b-2c'-3d)}f(-b'-2c'-3d) \\
&\quad \overline{f(-b-2c-3d')}f(-b'-2c-3d')f(-b-2c'-3d')\overline{f(-b'-2c'-3d')}
\end{aligned}
$$

where all the inequalities are by Cauchy-Schwarz. □

Example 4.17 Let M be an $\mathbb{F}_5^{n\times n}$ be a symmetric matrix. Then $f(x) = e(x^T Mx/5)$ satisfies $\|f\|_{U^3} = 1$.

Theorem 4.18 (U^3 Inverse Theorem) Let $f : \mathbb{F}_5^n \to \mathbb{C}$ satisfy $\|f\|_{L^\infty(\mathbb{F}_5^n)} \le 1$ and $\|f\|_{U^3(\mathbb{F}_5^n)} \ge \delta$ for some $\delta > 0$. Then there exists a symmetric matrix $M \in \mathbb{F}_5^{n\times n}$ and $b \in \mathbb{F}_5^n$ such that

$$
\left|\mathbb{E}_x f(x)e\left(x^T Mx + b^T x/p\right)\right| \ge c(\delta),
$$

where $c(\delta)$ is a polynomial in δ. In other words, $|\langle f, \varphi\rangle| \ge c(\delta)$ for $\varphi(x) = e\left(x^T Mx + b^T x/p\right)$, and we say "$f$ correlates with a quadratic phase function".

Proof sketch. We have $\|f\|_{U^3}^8 = \mathbb{E}_h \|\Delta_h f\|_{U^2}^4$ where $\Delta_h f(x) = f(x)\overline{f(x+h)}$.

1. Weak linearity: if $\|f\|_{U^3}^8 \ge \delta^8$, then for at least a $\delta^8/2$-proportion of $h \in \mathbb{F}_5^n$, $\delta^8/2 \le \|\Delta_h f\|_{U^2}^4 \le \left\|\widehat{\Delta_h f}\right\|_{\ell^\infty}^2$. So for each such $h \in \mathbb{F}_5^n$, there exists t_h such that $\left|\widehat{\Delta_h}(t_h)\right|^2 \ge \delta^8/2$. We have

   Proposition 4.19 Let $f : \mathbb{F}_5^n \to \mathbb{C}$ satisfy $\|f\|_{L^\infty(\mathbb{F}_5^n)} \le 1$ and $\|f\|_{U^3(\mathbb{F}_5^n)} \ge \delta$ for some $\delta > 0$. Suppose $|\mathbb{F}_5^n| = \Omega_\delta(1)$. Then there exists $S \subseteq \mathbb{F}_5^n$ with $|S| = \Omega_\delta(|\mathbb{F}_5^n|)$ and a function $\varphi : S \to \widehat{\mathbb{F}}_5^n$ such that:
   - $\left|\widehat{\Delta_h f}(\varphi(h))\right| = \Omega_\delta(1)$, and
   - There are at least $\Omega_\delta\left(|\mathbb{F}_5^n|^3\right)$ quadruples $(s_1, s_2, s_3, s_4) \in S^4$ such that $s_1 + s_2 = s_3 + s_4$ and $\varphi(s_1) + \varphi(s_2) = \varphi(s_3) + \varphi(s_4)$.
2. Strong linearity. If S and φ are as above, then there is a linear function $\psi : \mathbb{F}_5^n \to \widehat{\mathbb{F}}_5^n$ which coincides with φ for many elements of S. We have

   Proposition 4.20 Let S and φ be as given by above proposition. Then there exists a $M \in \mathbb{F}_5^{n\times n}$ and $b \in \mathbb{F}_5^n$ such that $\psi : \mathbb{F}_5^n \to \widehat{\mathbb{F}}_5^n$, $\psi(x) = Mx + b$ satisfies $\psi(x) = \varphi(x)$ for $\Omega_\delta(|\mathbb{F}_5^n|)$ elements $x \in S$.

Proof. Consider the graph of φ: $\Gamma = \{(h, \varphi(h) : h \in S\} \subseteq \mathbb{F}_5^n \times \widehat{\mathbb{F}}_5^n$. By above proposition, Γ has $\Omega_\delta(|\mathbb{F}_5^n|)$ additive quadruples. By Balog-Szemeredi-Gowers, there exists $\Gamma' \subseteq \Gamma$ with $|\Gamma'| = \Omega_\delta(|\Gamma|) = \Omega_\delta(|\mathbb{F}_5^n|)$ and $|\Gamma' + \Gamma'| = O_\delta(|\Gamma'|)$. Define $S' \subseteq S$ by $\Gamma' = \{(h, \varphi(h)) : h \in S'\}$. Note that $|S'| = \Omega_\delta(|\mathbb{F}_5^n|)$. By Freiman-Ruzsa applied to $\Gamma' \subseteq \mathbb{F}_5^n \times \widehat{\mathbb{F}}_5^n$, there exists a subspace $H \leq \mathbb{F}_5^n \times \widehat{\mathbb{F}}_5^n$ with $|H| = O_\delta(|\Gamma'|) = O_\delta(|\mathbb{F}_5^n|)$ such that $\Gamma' \subseteq H$.

Denote by $\pi : \mathbb{F}_5^n \times \widehat{\mathbb{F}}_5^n$ the projection onto the first n coordinates. By construction, $\pi(H) \supseteq S'$. Moreover, since $|S'| = \Omega_\delta(|\mathbb{F}_5^n|)$, we have

$$|\ker(\pi|_H)| = \frac{|H|}{|\mathrm{im}(\pi|_H)|} \leq \frac{O_\delta(|\mathbb{F}_5^n|)}{|S'|} = O_\delta(1).$$

We may thus partition H into $O_\delta(1)$ cosets of some subspace H^* such that $\pi|_H$ is injective on each coset. By averaging, there exists a coset $x + H^*$ such that $|\Gamma' \cap (x + H^*)| = \Omega_\delta(|\Gamma'|) = \Omega_\delta(|\mathbb{F}_5^n|)$.

Set $\Gamma'' = \Gamma' \cap (x + H^*)$ and define S'' accordingly. Now $\pi|_{x+H^*}$ is injective and surjective onto $V := \mathrm{im}(\pi|_{x+H^*})$. This means there is an affine-linear map $\psi : V \to \widehat{\mathbb{F}}_5^n$ such that $(h, \psi(h)) \in \Gamma'$ for all $h \in S''$. □

3. Symmetry argument.
4. Integration step.

Theorem 4.21 (Szemeredi's Theorem for 4-APs) Let $A \subseteq \mathbb{F}_5^n$ be a set containing no non-trivial 4-APs. Then $|A| = O(5^n)$.

Proof. Idea: by above proposition with $f = f_A = \mathbb{1}_A - \alpha$,

$$T_4(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) - \alpha^4 = T_4(f_A, f_A, f_A, f_A) + 14 \text{ other terms},$$

in which between one and three of the inputs are equal to f_A. These are controlled by $\|f_A\|_{U^2(\mathbb{F}_5^n)} \leq \|f_A\|_{U^3(\mathbb{F}_5^n)}$, whence

$$\left|T_4(\mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A, \mathbb{1}_A) - \alpha^4\right| \leq 15\|f_A\|_{U^3(\mathbb{F}_5^n)}$$

So if A contains no non-trivial 4-APs and $5^n > 2\alpha^{-3}$, then $\|f_A\|_{U^3(\mathbb{F}_5^n)} \geq \frac{\alpha^4}{30}$. What can we say about functions with large U^3 norm? □