Contents

1. The Khinchin axioms for entropy

Note all random variables we deal with will be discrete, unless otherwise stated. We use $\log = \log_2$.

1.1. Entropy axioms

Definition 1.1 The **entropy** of a discrete random variable X is a quantity $H(X)$ that takes real values and satisfies the **Khinchin axioms**: Normalisation, Invariance, Extendability, Maximality, Continuity and Additivity.

Axiom 1.2 (Normalisation) If X is uniform on $\{0, 1\}$ (i.e. $X \sim \text{Bern}(1/2)$), then $H(X) = 1$.

Axiom 1.3 (Invariance) If $Y = f(X)$ for some bijection f, then $H(Y) = H(X)$.

Axiom 1.4 (Extendability) If X takes values on a set A, B is disjoint from A, Y takes values in $A \sqcup B$, and for all $a \in A$, $\mathbb{P}(Y = a) = \mathbb{P}(X = a)$, then $H(Y) = H(X)$.

Axiom 1.5 (Maximality) If X takes values in a finite set A and Y is uniformly distributed in A, then $H(X) \leq H(Y)$.

Definition 1.6 The **total variance distance** between X and Y is

$$\sup_E |\mathbb{P}(X \in E) - \mathbb{P}(Y \in E)|.$$

Axiom 1.7 (Continuity) H depends continuously on X (with respect to total variation distance).

Definition 1.8 Let X and Y be random variables. The **conditional entropy** of X given Y is

$$H(X \mid Y) := \sum_y \mathbb{P}(Y = y) H(X \mid Y = y).$$

Axiom 1.9 (Additivity) $H(X, Y) := H((X, Y)) = H(Y) + H(X \mid Y)$.

1.2. Properties of entropy

Lemma 1.10 If X and Y are independent, then $H(X, Y) = H(X) + H(Y)$.

Proof (Hints). Straightforward. □

Proof. $H(X \mid Y) = \sum_y \mathbb{P}(Y = y) H(X \mid Y = y)$ Since X and Y are independent, the distribution of X is unaffected by knowing Y, so $H(X \mid Y = y) = H(X)$ for all y, which gives the result. (Note we have implicitly used Invariance here). □

Corollary 1.11 If $X_1, ..., X_n$ are independent, then

$$H(X_1, ..., X_n) = H(X_1) + \cdots + H(X_n).$$

Proof (Hints). Straightforward. □

Proof. By Lemma 1.10 and induction. □

Lemma 1.12 (Chain Rule) Let $X_1, ..., X_n$ be RVs. Then

$$H(X_1, ..., X_n) = H(X_1) + H(X_2 \mid X_1) + H(X_3 \mid X_1, X_2) + \cdots + H(X_n \mid X_1, ..., X_{n-1}).$$

Proof (Hints). Straightforward. □

Proof. The case $n = 2$ is Additivity. In general,

$$H(X_1, ..., X_n) = H(X_1, ..., X_{n-1}) + H(X_n \mid X_1, ..., X_{n-1}),$$

so the result follows by induction. □

Lemma 1.13 Let X and Y be RVs. If $Y = f(X)$, then $H(X, Y) = H(X)$. Also, $H(Z \mid X, Y) = H(Z \mid X)$.

Proof (Hints). Consider an appropriate bijection. □

Proof. The map $g : x \mapsto (x, f(x))$ is a bijection, and $(X, Y) = g(X)$, so the first statement follows from Invariance. Also,

$$\begin{aligned} H(Z \mid X, Y) &= H(Z, X, Y) - H(X, Y) \quad \text{by additivity} \\ &= H(Z, X) - H(X) \quad \text{by first part} \\ &= H(Z \mid X) \quad \text{by additivity} \end{aligned}$$

□

Lemma 1.14 If X takes only one value, then $H(X) = 0$.

Proof (Hints). Use that X and X are independent. □

Proof. X and X are independent (verify). So by Lemma 1.10, $H(X, X) = 2H(X)$. But by Invariance, $H(X, X) = H(X)$. So $H(X) = 0$. □

Proposition 1.15 If X is uniformly distributed on a set of size 2^n, then $H(X) = n$.

Proof (Hints). Straightforward. □

Proof. Let $X_1, ..., X_n$ be independent RVs, uniformly distributed on $\{0, 1\}$. By Corollary 1.11 and Normalisation, $H(X_1, ..., X_n) = n$. So the result follows by Invariance. □

Proposition 1.16 If X is uniformly distributed on a set A of size n, then $H(X) = \log n$.

Proof (Hints). Straightforward. □

Proof. Let $r \in \mathbb{N}$ and let $X_1, ..., X_r$ be independent copies of X. Then $(X_1, ..., X_r)$ is uniform on A^r, and $H(X_1, ..., X_r) = rH(X)$. Now pick k such that $2^k \le n^r \le 2^{k+1}$. Then by Proposition 1.15, Invariance and Maximality, $k \le rH(X) \le k + 1$. So $\frac{k}{r} \le \log n \le \frac{k+1}{r}$ and $\frac{k}{r} \le H(X) \le \frac{k+1}{r}$ for all $r \in \mathbb{N}$. So $H(X) = \log n$, as claimed. □

Theorem 1.17 (Khinchin) If H satisfies the Khinchin axioms and X takes values in a finite set A, then

$$H(X) = \sum_{a \in A} p_a \log(1/p_a) = \mathbb{E}\left[\log \frac{1}{P_X(X)}\right],$$

where $p_a = \mathbb{P}(X = a)$.

Proof (Hints).

- Explain why it is enough to prove for when the p_a are rational.
- Pick $n \in \mathbb{N}$ such that $p_a = \frac{m_a}{n}$, $m_a \in \mathbb{N}_0$. Let Z be uniform on $[n]$. Let $\{E_a : a \in A\}$ be a partition of $[n]$ into sets with $|E_a| = m_a$.

□

Proof. First we do the case where all $p_a \in \mathbb{Q}$. Pick $n \in \mathbb{N}$ such that $p_a = \frac{m_a}{n}$, $m_a \in \mathbb{N}_0$. Let Z be uniform on $[n]$. Let $\{E_a : a \in A\}$ be a partition of $[n]$ into sets with $|E_a| = m_a$. By Invariance, we may assume that $X = a \Leftrightarrow Z \in E_a$. Then

$$\begin{aligned} \log n = H(Z) = H(Z, X) &= H(X) + H(Z \mid X) \\ &= H(X) + \sum_{a \in A} p_a H(Z \mid X = a) \\ &= H(X) + \sum_{a \in A} p_a \log m_a \\ &= H(X) + \sum_{a \in A} p_a (\log p_a + \log n) \\ &= H(X) + \sum_{a \in A} p_a \log p_a + \log n. \end{aligned}$$

Hence $H(X) = -\sum_{a \in A} p_a \log p_a$.

The general result follows by Continuity. □

Corollary 1.18 Let X and Y be random variables. Then $0 \le H(X)$ and $0 \le H(X \mid Y)$.

Proof (Hints). Trivial. □

Proof. Immediate consequence of Khinchin. □

Corollary 1.19 If $Y = f(X)$, then $H(Y) \le H(X)$.

Proof (Hints). Straightforward. □

Proof. $H(X) = H(X, Y) = H(Y) + H(X \mid Y)$. But $H(X \mid Y) \ge 0$. □

Proposition 1.20 (Subadditivity) Let X and Y be RVs. Then $H(X, Y) \le H(X) + H(Y)$.

Proof (Hints).

- Let $p_{ab} = \mathbb{P}(X = a, Y = b)$. Explain why it is enough to show for the case when the p_{ab} are rational.
- Pick n such that $p_{ab} = m_{ab}/n$ with each $m_{ab} \in \mathbb{N}_0$. Partition $[n]$ into sets E_{ab} of size m_{ab}. Let Z be uniform on $[n]$.
- Show that if X (or Y) is uniform, then $H(X \mid Y) \le H(X)$ and $H(X, Y) \le H(X) + H(Y)$.
- Let $E_b = \cup_a E_{ab}$ for each b. So $Y = b$ iff $Z = E_b$. Now define an RV W as follows: if $Y = b$, then W is uniformly distributed in E_b. Use conditional independence to conclude the result.

□

Proof. Note that for any two RVs X, Y,

$$\begin{aligned} &H(X,Y) \leq H(X) + H(Y) \\ \iff &H(X \mid Y) \leq H(X) \\ \iff &H(Y \mid X) \leq H(Y) \end{aligned}$$

by Additivity. Next, observe that $H(X \mid Y) \leq H(X)$ if X is uniform on a finite set, since $H(X \mid Y) = \sum_y \mathbb{P}(Y = y) H(X \mid Y = y) \leq \sum_y \mathbb{P}(Y = y) H(X) = H(X)$ by Maximality. By the above equivalence, we also have $H(X \mid Y) \leq H(X)$ if Y is uniform on a finite set. Now let $p_{ab} = \mathbb{P}(X = a, Y = b)$, and assume that all p_{ab} are rational. Pick n such that $p_{ab} = m_{ab}/n$ with each $m_{ab} \in \mathbb{N}_0$. Partition $[n]$ into sets E_{ab} of size m_{ab}. Let Z be uniform on $[n]$. WLOG (by Invariance), $(X, Y) = (a, b)$ iff $Z \in E_{ab}$.

Let $E_b = \cup_a E_{ab}$ for each b. So $Y = b$ iff $Z = E_b$. Now define an RV W as follows: if $Y = b$, then $W \in E_b$, but then W is uniformly distributed in E_b and independent of X (and Z). So W and X are conditionally independent given Y, and W is uniform on $[n]$. Then $H(X \mid Y) = H(X \mid Y, W) = H(X \mid W)$ by conditional independence and by Lemma 1.13 (since W determines Y). Since W is uniform, $H(X \mid W) \leq H(X)$.

The general result follows by Continuity. □

Corollary 1.21 $H(X) \geq 0$ for any X.

Proof (Hints). (Without using the formula) straightforward. □

Proof. (Without using the formula). By subadditivity, $H(X \mid X) \leq H(X)$. But $H(X \mid X) = 0$. □

Corollary 1.22 Let $X_1, \dots, X_n$ be RVs. Then

$$H(X_1, \dots, X_n) \leq H(X_1) + \cdots + H(X_n).$$

Proof (Hints). Trivial. □

Proof. Trivial by induction. □

Proposition 1.23 (Submodularity) Let X, Y, Z be RVs. Then

$$H(X \mid Y, Z) \leq H(X \mid Z).$$

Proof (Hints). Use that $H(X \mid Y, Z = z) \leq H(Z \mid Z = z)$. □

Proof. $H(X \mid Y, Z) = \sum_z \mathbb{P}(Z = z) H(X \mid Y, Z = z) \leq \sum_z \mathbb{P}(Z = z) H(X \mid Z = z) = H(X \mid Z)$. □

Remark 1.24 Submodularity can be expressed in several equivalent ways. Expanding using Additivity gives

$$H(X, Y, Z) - H(Y, Z) \leq H(X, Z) - H(Z)$$

and

$$H(X, Y, Z) \leq H(X, Z) + H(Y, Z) - H(Z)$$

and

$$H(X, Y, Z) + H(Z) \leq H(X, Z) + H(Y, Z).$$

Lemma 1.25 Let X, Y, Z be RVs with $Z = f(Y)$. Then $H(X \mid Y) \leq H(X \mid Z)$.

Proof (Hints). Straightforward. □

Proof. We have

$$\begin{aligned} H(X \mid Y) = H(X, Y) - H(Y) &= H(X, Y, Z) - H(Y, Z) \\ &\leq H(X, Z) - H(Z) = H(X \mid Z) \end{aligned}$$

by Submodularity. □

Lemma 1.26 Let X, Y, Z be RVs with $Z = f(X) = g(Y)$. Then

$$H(X, Y) + H(Z) \leq H(X) + H(Y).$$

Proof (Hints). Straightforward. □

Proof. By Submodularity, we have $H(X, Y, Z) + H(Z) \leq H(X, Z) + H(Y, Z)$, which implies the result, since Z depends on X and Y. □

Lemma 1.27 Let X be an RV taking values in a finite set A and let Y be uniform on A. If $H(X) = H(Y)$, then X is uniform.

Proof (Hints). Use Jensen's inequality. □

Proof. Let $p_a = \mathbb{P}(X = a)$. Then

$$H(X) = \sum_{a \in A} p_a \log(1/p_a) = |A| \cdot \mathbb{E}_{a \in A} p_a \log\left(\frac{1}{p_a}\right).$$

The function $x \mapsto x \log(1/x)$ is concave on $[0, 1]$. So by Jensen's inequality,

$$H(X) \leq |A| \cdot (\mathbb{E}_{a \in A} p_a) \cdot \log\left(\frac{1}{\mathbb{E}_{a \in A} p_a}\right) = \log|A| = H(Y),$$

with equality iff $a \mapsto p_a$ is constant, i.e. X is uniform. □

Corollary 1.28 If $H(X, Y) = H(X) + H(Y)$, then X and Y are independent.

Proof (Hints). Go through the proof of Subadditivity and check when equality holds. □

Proof. We go through the proof of subadditivity and check when equality holds. Suppose that X is uniform on A. Then

$$H(X \mid Y) = \sum_y \mathbb{P}(Y = y) H(X \mid Y = y) \leq H(X),$$

with equality iff $H(X \mid Y = y)$ is uniform on A for all y (by Lemma 1.27), which implies that X and Y are independent.

At the last stage of the proof, we said $H(X \mid Y) = H(X \mid Y, W) = H(X \mid W) \le H(X)$, where W was uniform. So equality holds only if X and W are independent, which implies (since Y depends on W), that X and Y are independent. $\square$

Definition 1.29 Let X and Y be RVs. The **mutual information**

$$\begin{aligned} I(X : Y) &:= H(X) + H(Y) - H(X, Y) \\ &= H(X) - H(X \mid Y) \\ &= H(Y) - H(Y \mid X). \end{aligned}$$

Remark 1.30 Subadditivity is equivalent to the statement that $I(X : Y) \ge 0$, and Corollary 1.28 implies that $I(X : Y) = 0$ iff X and Y are independent.

Note that $H(X, Y) = H(X) + H(Y) - I(X : Y)$ (note the similarity to the inclusion-exclusion formula for two sets).

Definition 1.31 Let X, Y, Z be RVs. The **conditional mutual information** of X and Y given Z is

$$\begin{aligned} I(X : Y \mid Z) &:= \sum_z \mathbb{P}(Z = z) I(X \mid Z = z : Y \mid Z = z) \\ &= \sum_z \mathbb{P}(Z = z)(H(X \mid Z = z) + H(Y \mid Z = z) - H(X, Y \mid Z = z)) \\ &= H(X \mid Z) + H(Y \mid Z) - H(X, Y \mid Z) \\ &= H(X, Z) + H(Y, Z) - H(X, Y, Z) - H(Z). \end{aligned}$$

Submodularity is equivalent to the statement that $I(X : Y \mid Z) \ge 0$.

2. A special case of Sidorenko's conjecture

Definition 2.1 Let G be a bipartite graph with (finite) vertex sets X and Y and density α (defined to be $\frac{|E(G)|}{|X| \cdot |Y|}$). Let H be another (think of it as small) bipartite graph with vertex sets U and V and m edges. Now let $\varphi : U \to X$ and $\psi : V \to Y$. We say that (φ, ψ) is a **homomorphism** if $\varphi(x)\varphi(y) \in E(G)$ for every edge $xy \in E(H)$.

Conjecture 2.2 (Sidorenko's Conjecture) For every G, H, for random $\varphi : U \to X$, $\psi : V \to Y$,

$$\mathbb{P}((\varphi, \psi) \text{ is a homomorphism}) \ge \alpha^m.$$

Remark 2.3 Sidorenko's Conjecture is not hard to prove when H is the complete bipartite graph $K_{r,s}$ (the case $K_{2,2}$ can be proved using Cauchy-Schwarz: exercise).

Theorem 2.4 Sidorenko's Conjecture is true if H is a path of length 3.

Proof (Hints).

- Let (X_1, Y_1) be a random edge of G (with $X_1 \in X$, $Y_1 \in Y$). Now let X_2 be a random neighbour of Y_1 and Y_2 be a random neighbour of X_2. Explain why it suffices to prove that $H(X_1, Y_1, X_2, Y_2) \geq \log(\alpha^3 m^2 n^2)$.
- Find an equivalent way of choosing a uniformly random edge (X_1, Y_1) of G (in terms of vertices). Use this to reaosn that $X_2 Y_1$ and $X_2 Y_2$ are uniformly random in $E(G)$.
- Find the lower bound for $H(X_1, Y_1, X_2, Y_2)$ using the Chain Rule and Maximality.

□

Proof. We want to show that if G is a bipartite graph of density α with vertex sets X, Y of size m and n, and we choose $x_1, x_2 \in X$, $y_1, y_2 \in Y$ independently at random, then $\mathbb{P}(x_1 y_1, y_1 x_2, x_2 y_2 \in E(G)) \geq \alpha^3$.

It would be enough to let P be a path of length 3 chosen uniformly at random and show that $H(P) \geq \log(\alpha^3 m^2 n^2)$ (by Proposition 1.16). Instead, we shall define a different RV taking values in the set of all paths of length 3 (including degenerate paths). To do this, let (X_1, Y_1) be a random edge of G (with $X_1 \in X$, $Y_1 \in Y$). Now let X_2 be a random neighbour of Y_1 and Y_2 be a random neighbour of X_2. It will be enough to prove that

$$H(X_1, Y_1, X_2, Y_2) \geq \log(\alpha^3 m^2 n^2).$$

We can choose X_1, Y_1 in three equivalent ways:
1. Pick an edge uniformly from all edges
2. Pick a vertex x with probability proportional to its degree $\deg(x)$, and then a random neighbour Y of x.
3. Same as above with x and y exchanged.

By the equivalence, it follows that $Y_1 = y$ with probability $\deg(y)/|E(G)|$, so $X_2 Y_1$ is uniform in $E(G)$, so $X_2 = x'$ with probability $d(x')/|E(G)|$, so $X_2 Y_2$ is uniform in $E(G)$.

Let U_A be the uniform distribution on A. Therefore, by the Chain Rule,

$$\begin{aligned}
H(X_1, Y_1, X_2, Y_2) &= H(X_1) + H(Y_1 \mid X_1) + H(X_2 \mid X_1, Y_1) + H(Y_2 \mid X_1, Y_1, X_2) \\
&= H(X_1) + H(Y_1 \mid X_1) + H(X_2 \mid Y_1) + H(Y_2 \mid X_2) \\
&= H(X_1) + H(X_1, Y_1) - H(X_1) + H(X_2, Y_1) - H(Y_1) + H(X_2, Y_2) - H(Y_2) \\
&= 3H\left(U_{E(G)}\right) - H(Y_1) - H(X_2) \\
&\geq 3H\left(U_{E(G)}\right) - H(U_Y) - H(U_X) \\
&= 3\log(\alpha m n) - \log n - \log m \\
&= \log(\alpha^3 m^2 n^2).
\end{aligned}$$

So we are done, by Maximality. Alternative finish to the proof: let X', Y' be uniform in X, Y and independent of each other and X_1, Y_1, X_2, Y_2. Then by the above inequality and Corollary 1.11,

$$H(X_1, Y_1, X_2, Y_2, X', Y') = H(X_1, Y_1, X_2, Y_2) + H(U_X) + H(U_Y)$$

$$\geq 3H\left(U_{E(G)}\right).$$

So by Maximality, the number of paths of length 3 times $|X|$ times $|Y|$ is $\geq |E(G)|^3$. $\square$

3. Brigner's theorem

Definition 3.1 Let A be an $n \times n$ matrix over $\mathbb{R}$. The **permanent** of A is

$$\operatorname{per}(A) := \sum_{\sigma \in S_n} \prod_{i=1}^{n} A_{i\sigma(i)},$$

i.e. "the determinant without the signs".

Proposition 3.2 Let G be a bipartite graph with vertex sets X, Y of size n. Given $(x, y) \in X \times Y$, let

$$A_{xy} = \begin{cases} 1 & \text{if } xy \in E(G) \\ 0 & \text{if } xy \notin E(G) \end{cases},$$

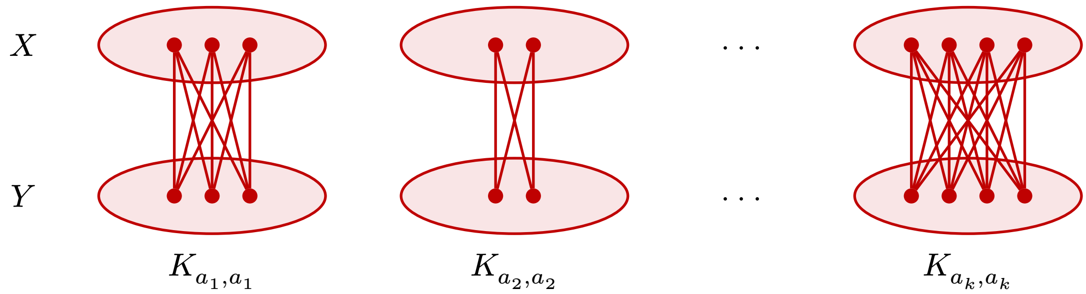
i.e. A is the bipartite adjacency matrix of G. Then $\operatorname{per}(A)$ is the number of perfect matchings in G. (Note that $\operatorname{per}(A)$ is well-defined as it is invariant under reordering of the vertices.)

Proof (Hints). Straightforward. $\square$

Proof. Each (perfect) matching corresponds to a bijection $\sigma : X \to Y$ such that $x\sigma(x) \in E(G)$ for all $x \in X$. $\sigma \in S_n$ contributes 1 to the sum iff $x\sigma(x)$ is an edge of G for all $x \in X$ (i.e. iff σ corresponds to a perfect matching), and 0 otherwise. $\square$

Bregman's theorem concerns how large $\operatorname{per}(A)$ can be if A is a $0, 1$ matrix and the sum of the entries in the i-th row is d_i (i.e. if the degree of $x_i \in X$ is d_i).

Example 3.3 Let G be a disjoint union of K_{a_i,a_i}'s, $i = 1, ..., k$, with $a_1 + \cdots + a_k = n$. Then the number of perfect matchings in G is $\prod_{i=1}^{k} a_i!$.



Theorem 3.4 (Bregman) Let G be a bipartite graph with vertex sets X, Y of size n. Then the number of perfect matchings in G is at most

$$\prod_{x \in X} (\deg(x)!)^{1/\deg(x)}.$$

Proof (Hints).

- For an enumeration $x_1, \ldots, x_n$ of X and random matching (a bijection) σ, show that $H(\sigma) \le \log \deg(x_1) + \mathbb{E}_\sigma \log \deg^\sigma_{x_1}(x_2) + \cdots + \mathbb{E}_\sigma \log \deg^\sigma_{x_1,\ldots,x_{n-1}}(x_n)$ (find a suitable expression for $\deg^\sigma_{x_1,\ldots,x_{i-1}}(x_i)$).
- Find another expression for $\deg^\sigma_{x_1,\ldots,x_{i-1}}(x_i)$ in terms of $\deg(x)$.
- Show that the average of $\log \deg^\sigma_{x_1,\ldots,x_{i-1}}(x_i)$ is $\frac{1}{d(x)}(\log(d(x)!))$.

□

Proof (by Radhakrishnan). Each (perfect) matching corresponds to a bijection $\sigma : X \to Y$ such that $x\sigma(x) \in E(G)$ for all $x \in X$. Let σ be chosen uniformly from all such bijections. Then by the Chain Rule,

$$\begin{aligned} H(\sigma) &= H(\sigma(x_1), \ldots, \sigma(x_n)) \\ &= H(\sigma(x_1)) + H(\sigma(x_2) \mid \sigma(x_1)) + \cdots + H(\sigma(x_n) \mid \sigma(x_1), \ldots, \sigma(x_{n-1})), \end{aligned}$$

where $x_1, \ldots, x_n$ is some enumeration of X. We have $H(\sigma(x_1)) \le \log \deg(x_1)$ by Maximality, and

$$H(\sigma(x_2) \mid \sigma(x_1)) \le \mathbb{E}_\sigma \log \deg^\sigma_{x_1}(x_2),$$

where $\deg^\sigma_{x_1}(x_2) = |N(x_2) \setminus \{\sigma(x_1)\}|$, by the definition of conditional entropy and Maximality. In general,

$$H(\sigma(x_i) \mid \sigma(x_1), \ldots, \sigma(x_{i-1})) \le \mathbb{E}_\sigma \log \deg^\sigma_{x_1,\ldots,x_{i-1}}(x_i),$$

where $\deg^\sigma_{x_1,\ldots,x_{i-1}}(x_i) = |N(x_i) \setminus \{\sigma(x_1), \ldots, \sigma(x_{i-1})\}|$.

Key idea: we now regard $x_1, \ldots, x_n$ as a *random* enumeration of X and take the average. For each $x \in X$, define the **contribution** of x to be $\log\left(d^\sigma_{x_1,\ldots,x_{i-1}}(x_i)\right)$, where $x_i = x$. We shall now fix σ and $x \in X$. Let the neighbours of x be $y_1, \ldots, y_k$. Then one of the y_j will be $\sigma(x)$, say y_h. Then $d^\sigma_{x_1,\ldots,x_{i-1}}(x_i)$ (given that $x_i = x$) is

$$d(x) - \left|\left\{j : \sigma^{-1}(y_j) \text{ comes earlier than } x = \sigma^{-1}(y_h)\right\}\right|.$$

All positions of $\sigma^{-1}(y_h)$ are equally likely, so the average contribution of x is

$$\begin{aligned} &\frac{1}{d(x)}(\log d(x) + \log(d(x)-1) + \cdots + \log(1)) \\ &= \frac{1}{d(x)} \log d(x)!. \end{aligned}$$

By linearity of expectation,

$$H(\sigma) \le \sum_{x \in X} \frac{1}{d(x)} \log(d(x)!)$$

So the number of matchings is at most $\prod_{x \in X} (d(x)!)^{1/d(x)}$. □

Definition 3.5 Let G be a graph with $2n$ vertices. A **1-factor** in G is a collection of n disjoint edges.

Theorem 3.6 (Kahn-Lovasz) Let G be a graph with $2n$ vertices. Then the number of 1-factors in G is at most

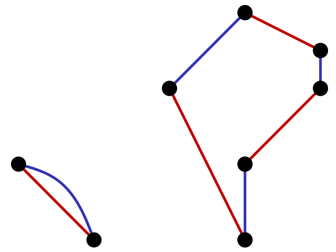
$$\prod_{x \in V(G)} (d(x)!)^{1/2d(x)}.$$

Proof (Hints).

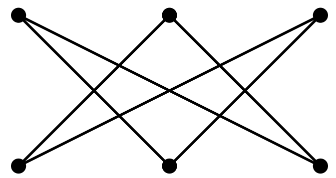
- Let M be the set of 1-factors of G and let (M_1, M_2) be a uniformly random element of $M \times M$.
- Given a cover of G by M_1 and M_2, find an expression for the number of pairs (M_1', M_2') that could give rise to it, in terms of the number of even cycles.
- Let G_2 be the bipartite graph with two vertex sets V_1, V_2, which are both copies of $V(G)$. Join $x \in V_1$ to $y \in V_2$ iff $xy \in E(G)$.
- Explain why each perfect matching of G_2 gives a cover of $V(G)$ by isolated vertices, edges and cycles, and find an expression for the number of such perfect matchings that could give rise to it.

$\square$

Proof (by Alon, Friedman). Let M be the set of 1-factors of G and let (M_1, M_2) be a uniformly random element of $M \times M$. For each M_1, M_2, the union $M_1 \cup M_2$ is a collection of disjoint edges and even cycles that covers all the vertices of G.



Call such a union a **cover of G by edges and even cycles**. If we are given such a cover, then the number of pairs (M_1, M_2) that could give rise to it is 2^k, where k is the number of even cycles. Now let's build a bipartite graph G_2 out of G. G_2 has two vertex sets V_1, V_2, which are both copies of $V(G)$. Join $x \in V_1$ to $y \in V_2$ iff $xy \in E(G)$.



G_2 if G is the triangle graph

By Bregman, the number of perfect matchings in G_2 is at most $\prod_{x \in V(G)} (d(x)!)^{1/d(x)}$. Each matching gives a permutation σ of $V(G)$ such that $x\sigma(x) \in E(G)$ for all $x \in V(G)$. Each such σ has a cycle decomposition, and each cycle gives a cycle in G. So σ gives a cover of $V(G)$ by isolated vertices, edges and cycles (not necessarily all even). Given such a cover with k cycles, each cycle can be directed in two ways, so the number of σ that give rise to it is $= 2^k$. So there is an injection from $M \times M$ to the set of matchings of G_2, since every cover by edges and and even cycles is a cover by vertices, edges and cycles. So $|M|^2 \le \prod_{x \in V(G)} (d(x)!)^{1/d(x)}$. □

4. Shearer's lemma and applications

Notation 4.1 Given a random variable $X = (X_1, ..., X_n)$ and $A \subseteq [n]$, $A = \{a_1 < ... < a_k\}$, write X_A for the random variable $\left(X_{a_1}, ..., X_{a_k}\right)$.

Lemma 4.2 (Shearer) Let $X = (X_1, ..., X_n)$ be an RV and let $\mathcal{A}$ be a family of subsets of $[n]$ such that every $i \in [n]$ belongs to at least r of the sets $A \in \mathcal{A}$. Then

$$H(X_1, ..., X_n) \le \frac{1}{r} \sum_{A \in \mathcal{A}} H(X_A).$$

Proof (Hints). For each $a \in [n]$, write $X_{<a}$ for $(X_1, ..., X_{a-1})$. Show that $H(X_A) \ge \sum_{a \in A} H(X_a \mid X_{<a})$. □

Proof. For each $a \in [n]$, write $X_{<a}$ for $(X_1, ..., X_{a-1})$. For each $A \in \mathcal{A}$, $A = \{a_1 < \cdots < a_k\}$, by the Chain Rule and Submodularity,

$$\begin{aligned} H(X_A) &= H\left(X_{a_1}\right) + H\left(X_{a_2} \mid X_{a_1}\right) + \cdots + H\left(X_{a_k} \mid X_{a_1}, ..., X_{a_{k-1}}\right) \\ &\ge H\left(X_{a_1} \mid X_{<a_1}\right) + H\left(X_{a_2} \mid X_{<a_2}\right) + \cdots + H\left(X_{a_k} \mid X_{<a_k}\right) \\ &= \sum_{a \in A} H(X_a \mid X_{<a}). \end{aligned}$$

Therefore, $\sum_{A \in \mathcal{A}} H(X_A) \ge r \sum_{a=1}^{n} H(X_a \mid X_{<a}) = rH(X)$. □

Example 4.3 $H(X_1, X_2, X_3) \le \frac{1}{2}(H(X_1, X_2) + H(X_1, X_3) + H(X_2, X_3))$.

Lemma 4.4 Let $X = (X_1, ..., X_n)$ be an RV and let $A \subseteq [n]$ be a randomly chosen subset of $[n]$, according to some probability distribution. Suppose that for each $i \in [n]$, $\mathbb{P}(i \in A) \ge \mu$. Then

$$H(X) \le \mu^{-1} \cdot \mathbb{E}_A[H(X_A)].$$

Proof (Hints). Very similar to proof of Shearer. □

Proof. As in Shearer,

$$H(X_A) \ge \sum_{a \in A} H(X_a \mid X_{<a}).$$

So

$$\mathbb{E}_A[H(X_A)] \ge \mathbb{E}_A\left[\sum_{a \in A} H(X_a \mid X_{<a})\right] \ge \mu \cdot \sum_{a=1}^{n} H(X_a \mid X_{<a}) = \mu \cdot H(X).$$

□

Definition 4.5 Let $E \subseteq \mathbb{Z}^n$ and let $A \subseteq [n]$. Then we write $P_A E$, if $A = \{a_1, ..., a_k\}$, for the set of $u \in \mathbb{Z}^A$ such that there exists $v \in \mathbb{Z}^{[n]\setminus A}$ such that $[u, v] \in E$, where $[u, v]$ is u suitably intertwined with v.

Corollary 4.6 Let $E \subseteq \mathbb{Z}^n$ and let $\mathcal{A}$ be a family of subsets of $[n]$ such that every $i \in [n]$ is contained in at least r sets in $\mathcal{A}$. Then

$$|E| \leq \prod_{A \in \mathcal{A}} |P_A E|^{1/r}.$$

Proof (Hints). Straightforward. □

Proof. Let X be a uniformly random element of E. Then by Shearer,

$$\log|E| = H(X) \leq \frac{1}{r} \cdot \sum_{A \in \mathcal{A}} H(X_A).$$

But X_A takes values in $P_A E$, so $H(X_A) \leq \log|P_A E|$ by Maximality. Hence,

$$\log|E| \leq \frac{1}{r} \sum_{A \in \mathcal{A}} |P_A E|.$$

□

Corollary 4.7 (Discrete Loomis-Whitney Theorem) If $\mathcal{A} = \{[n] \setminus \{i\} : i = 1, ..., n\}$, we get

$$|E| \leq \prod_{i=1}^{n} \left|P_{[n]\setminus\{i\}} E\right|^{1/(n-1)}.$$

Theorem 4.8 Let G be a graph with m edges. Then G has at most $\frac{1}{6}(2m)^{3/2}$ triangles.

Remark 4.9 If $m = \binom{n}{2}$, then this bound is fairly sharp.

Proof (Hints). Consider a uniformly random triangle with an ordering on the vertices, and use Shearer. □

Proof. Let (X_1, X_2, X_3) be a random triple of vertices such that $X_1 X_2$, $X_1 X_3$ and $X_2 X_3$ are all edges (so pick a random triangle with an ordering of the vertices). Let t be the number of triangles in G. By Shearer,

$$\log(6t) = H(X_1, X_2, X_3) \leq \frac{1}{2}(H(X_1, X_2) + H(X_1, X_3) + H(X_2, X_3)).$$

Each $\left(X_i, X_j\right)$ (for $i \neq j$) is supported in the set of edges of G, given a direction, so $H\left(X_i, X_j\right) \leq \log(2m)$ by Maximality. □

Definition 4.10 Let V be a set of size n and let $\mathcal{G}$ be a set of graphs, all with vertex set V. Then $\mathcal{G}$ is **Δ-intersecting** (triangle-intersecting) if $G_1 \cap G_2$ contains a triangle for all $G_1, G_2 \in \mathcal{G}$.

Theorem 4.11 If $|V| = n$, then a Δ-intersecting family of graphs with vertex set V has size at most $2^{\binom{n}{2}-2}$.

Proof (Hints).

- Let $\mathcal{G}$ be a Δ-intersecting family. View $G \in \mathcal{G}$ as a characteristic function from $V^{(2)}$ to $\{0,1\}$. Let $X = \left(X_e : e \in V^{(2)}\right)$ be chosen uniformly at random from $\mathcal{G}$.
- Let $G_R = K_R \cup K_{V\setminus R}$, explain why G_R is an intersecting family, use this to give upper bound on $|G_R|$.
- Give an expression for the probability that an edge e is in a random G_R. By considering X_{G_R} taking values in the above family, conclude.

$\square$

Proof. Let $\mathcal{G}$ be a Δ-intersecting family and let X be chosen uniformly at random from $\mathcal{G}$. We write $V^{(2)}$ for the set of (unordered) pairs of elements of V. We think of any $G \in \mathcal{G}$ as a characteristic function from $V^{(2)}$ to $\{0,1\}$. So $X = \left(X_e : e \in V^{(2)}\right)$, $X_e \in \{0,1\}$ (where we fix an ordering of $V^{(2)}$). For each $R \subseteq V$, let G_R be the graph $K_R \cup K_{V\setminus R}$. For each R, we shall look at the projection X_{G_R}, which we can think of as taking values in the set $\{G \cap G_R : G \in \mathcal{G}\} =: \mathcal{G}_R$.

Note that if $G_1, G_2 \in \mathcal{G}$, $R \subseteq [n]$, then $G_1 \cap G_2 \cap G_R \neq \emptyset$, since $G_1 \cap G_2$ contains a triangle, which must intersect G_R by the pigeonhole principle (the triangle contains 3 vertices, one of which is contained in one of the two components of G_R). Thus, $\mathcal{G}_R$ is an intersecting family, so has size at most $2^{|E(G_R)|-1}$. By Lemma 4.4,

$$H(X) \leq 2 \cdot \mathbb{E}_R\left[H\left(X_{G_R}\right)\right] \leq 2 \cdot \mathbb{E}_R[|E(G_R)| - 1] = 2 \cdot \left(\frac{1}{2}\binom{n}{2} - 1\right) = \binom{n}{2} - 2,$$

since each e belongs to G_R with probability $1/2$ (and so $\mathbb{E}_R[|E(G_R)|] = \frac{1}{2}\binom{n}{2}$). $\square$

Definition 4.12 Let G be a graph and let $A \subseteq V(G)$. The **edge-boundary** ∂A of A is the set of edges xy such that $x \in A$, $y \notin A$. If $G = \mathbb{Z}^n$ or $\{0,1\}^n$ and $i \in [n]$, the ***i*-th boundary** $\partial_i A$ is the set of edges $xy \in \partial A$ such that $x - y = \pm e_i$, i.e. $\partial_i A$ consists of edges in direction i.

Theorem 4.13 (Edge-isoperimetric Inequality in $\mathbb{Z}^n$) Let $A \subseteq \mathbb{Z}^n$ be a finite set. Then

$$|\partial A| \geq 2n \cdot |A|^{(n-1)/n}.$$

Proof (Hints). Use Discrete Loomis-Whitney Theorem and a suitable lower bound on $|\partial_i A|$. $\square$

Proof. By the Discrete Loomis-Whitney Theorem,

$$\begin{aligned}
|A| &\leq \prod_{i=1}^{n} \left|P_{[n]\setminus\{i\}} A\right|^{1/(n-1)} \\
&= \left(\prod_{i=1}^{n} \left|P_{[n]\setminus\{i\}} A\right|^{1/n}\right)^{n/(n-1)} \\
&\leq \left(\frac{1}{n}\sum_{i=1}^{n} \left|P_{[n]\setminus\{i\}} A\right|\right)^{n/(n-1)} \qquad \text{by AM-GM inequality}
\end{aligned}$$

But $|\partial_i A| \geq 2\left|P_{[n]\setminus\{i\}} A\right|$ since each fibre contributes at least 2. So

$$\begin{aligned} |A| &\leq \left(\frac{1}{2n} \sum_{i=1}^{n} |\partial_i A|\right)^{n/(n-1)} \\ &= \left(\frac{1}{2n} |\partial A|\right)^{n/(n-1)} \end{aligned}$$

□

Theorem 4.14 (Edge-isoperimetric Inequality in the Cube) Let $A \subseteq \{0,1\}^n$ (where we take usual graph on $\{0,1\}^n$). Then

$$|\partial A| \geq |A|(n - \log|A|).$$

Proof (Hints).

- Let $X = (X_1, ..., X_n)$ be a uniformly random element of A. Write $X_{\setminus i} = (X_1, ..., X_{i-1}, X_{i+1}, ..., X_n)$.
- Use Shearer to show that $\sum_{i=1}^{n} H\left(X_i \mid X_{\setminus i}\right) \leq H(X)$.
- What are the possible values of $\left|P_{[n]\setminus\{i\}}^{-1}(u)\right|$, and what is $H\left(X_i \mid X_{\setminus i} = u\right)$ in each case? How many u satisfy $\left|P_{[n]\setminus\{i\}}^{-1}(u)\right| = 1$? Use this to deduce an expression for $H\left(X_i \mid X_{\setminus i}\right)$.

□

Proof. Let X be a uniformly random element of A and write $X = (X_1, ..., X_n)$. Write $X_{\setminus i}$ for $(X_1, ..., X_{i-1}, X_{i+1}, ..., X_n)$. By Shearer,

$$\begin{aligned} H(X) &\leq \frac{1}{n-1} \sum_{i=1}^{n} H\left(X_{\setminus i}\right) \\ &= \frac{1}{n-1} \sum_{i=1}^{n} \left(H(X) - H\left(X_i \mid X_{\setminus i}\right)\right). \end{aligned}$$

Hence, $\sum_{i=1}^{n} H\left(X_i \mid X_{\setminus i}\right) \leq H(X)$. But

$$H\left(X_i \mid X_{\setminus i} = u\right) = \begin{cases} 1 \text{ if } \left|P_{[n]\setminus\{i\}}^{-1}(u)\right| = 2 \\ 0 \text{ if } \left|P_{[n]\setminus\{i\}}^{-1}(u)\right| = 1 \end{cases}$$

(Note that we always have $\left|P_{[n]\setminus\{i\}}^{-1}(u)\right| \in \{0,1,2\}$). The number of points of the second kind is $|\partial_i A|$. So

$$\begin{aligned} H\left(X_i \mid X_{\setminus i}\right) &= \sum_u \mathbb{P}\left(X_{\setminus i} = u\right) H\left(X_i \mid X_{\setminus i = u}\right) \\ &= \sum_{u \notin \partial_i A} \mathbb{P}\left(X_{\setminus i} = u\right) \\ &= 1 - \sum_{u \in \partial_i A} \mathbb{P}\left(X_{\setminus i} = u\right) \end{aligned}$$

$$= 1 - \frac{|\partial_i A|}{|A|}.$$

So

$$H(X) \geq \sum_{i=1}^{n} \left(1 - \frac{|\partial_i A|}{|A|}\right)$$
$$= n - \frac{|\partial A|}{|A|}.$$

Also, $H(X) = \log|A|$. So we are done. $\square$

Definition 4.15 Let $\mathcal{A}$ be a family of sets of size d. The **lower shadow** of $\mathcal{A}$ is

$$\partial\mathcal{A} = \{B : |B| = d - 1, \exists A \in \mathcal{A} \text{ s.t. } B \subseteq A\}.$$

Theorem 4.16 (Kruskal-Katona) If $|\mathcal{A}| = \binom{t}{d} = \frac{t(t-1)\cdots(t-d+1)}{d!}$ for some real number t, then

$$|\partial_i \mathcal{A}| \geq \binom{t}{d-1}.$$

Proof (Hints).

- Let $X = (X_1, ..., X_d)$ be a random ordering of the elements of a uniformly random $A \in \mathcal{A}$. Give an expression for $H(X)$.
- Explain why it is enough to show $H(X_1, ..., X_{d-1}) \geq \log\left((d-1)!\binom{t}{d-1}\right)$.
- Let $T \sim \mathrm{Bern}(p)$ be independent of $X_1, ..., X_{k-1}$, and given $X_1, ..., X_{k-1}$, let

$$X^* = \begin{cases} X_{k+1} & \text{if } T = 0 \\ X_k & \text{if } T = 1 \end{cases}.$$

- Show that $H(X_k \mid X_{<k}) \geq H(X^*, T \mid X_{\leq k}) = h(p) + pH(X_{k+1} \mid X_{\leq k})$, and so that $H(X_k \mid X_{<k}) \geq \log\left(2^{H(X_{k+1} \mid X_{\leq k})} + 1\right)$.
- Using the chain rule, show that $r + d - 1 \leq t$, and use this to conclude the desired bound on $H(X_{<d})$.

$\square$

Proof. Let $X = (X_1, ..., X_d)$ be a random ordering of the elements of a uniformly random $A \in \mathcal{A}$. Then $H(X) = \log(d!|A|) = \log\left(d!\binom{t}{d}\right)$. Note that $(X_1, ..., X_{d-1})$ is an ordering of the elements of some $B \in \partial_i A$, so

$$H(X_1, ..., X_{d-1}) \leq \log((d-1)!|\partial_i A|)$$

So it's enough to show $H(X_1, ..., X_{d-1}) \geq \log\left((d-1)!\binom{t}{d-1}\right)$. Also, $H(X) = H(X_1, ..., X_{d-1}) + H(X_d \mid X_1, ..., X_{d-1})$ and $H(X) = H(X_1) + H(X_2 \mid X_1) + \cdots + H(X_d \mid X_1, ..., X_{d-1})$. We would like an upper bound for $H(X_d \mid X_{<d})$. Our strategy will be to obtain a lower bound for $H(X_k \mid X_{<k})$ in terms of $H(X_{k+1} \mid X_{<k+1})$. We shall prove that $2^{H(X_k \mid X_{<k})} \geq 2^{H(X_{k+1} \mid X_{<k+1})} + 1$ for all k.

Let T be chosen independently of X. Let $T \sim \text{Bern}(1-p)$ ($T = 0$ with probability p, p is to be chosen later). Given $X_1, ..., X_{k-1}$, let

$$X^* = \begin{cases} X_{k+1} & \text{if } T = 0 \\ X_k & \text{if } T = 1 \end{cases}$$

Note that X_k and X_{k+1} have the same distribution (given $X_1, ..., X_{k-1}$), so X^* does as well. Then

$$\begin{aligned}
H(X_k \mid X_{<k}) &= H(X^* \mid X_{<k}) \text{ since } X_k \sim X^* \\
&\geq H(X^* \mid X_{\leq k}) \quad \text{by Submodularity} \\
&= H(X^*, T \mid X_{\leq k}) \quad \text{since } X_{\leq k} \text{ and } X^* \text{ determine } T \text{ (since } X_{k+1} \neq X_k) \\
&= H(T \mid X_{\leq k}) + H(X^* \mid T, X_{\leq k}) \quad \text{by Additivity} \\
&= H(T) + pH(X^* \mid X_{\leq k}, T = 0) + (1-p)H(X^* \mid X_{\leq k}, T = 1) \\
&= H(T) + pH(X_{k+1} \mid X_{\leq k}) + (1-p)H(X_k \mid X_{\leq k}) \\
&= h(p) + ps.
\end{aligned}$$

where $s = H(X_{k+1} \mid X_{\leq k})$ and $h(p) = p \log \frac{1}{p} + (1-p) \log \frac{1}{1-p}$. This is maximised when $p = \frac{2^s}{2^s+1}$. Then we get

$$\frac{2^s}{2^s+1}(\log(2^s+1) - \log(2^s)) + \frac{1}{2^s+1}(\log(2^s+1)) + \frac{s2^s}{2^s+1} = \log(2^s+1).$$

This proves the claim.

Let $r = 2^{H(X_d \mid X_{<d})}$. Then by the claim,

$$\begin{aligned}
H(X) &= H(X_1) + \cdots + H(X_d \mid X_{<d}) \\
&\geq \log(r+d-1) + \cdots + \log(r) \\
&= \log\left(\frac{(r+d-1)!}{(r-1)!}\right) = \log\left(d!\binom{r+d-1}{d}\right).
\end{aligned}$$

Since $H(X) = \log\left(d!\binom{t}{d}\right)$ is an increasing function (for $t \geq d$), it follows that $r + d - 1 \leq t$, i.e. $r \leq t + 1 - d$. It follows that

$$\begin{aligned}
H(X_{<d}) &= \log\left(d!\binom{t}{d}\right) - \log r \\
&\geq \log\left(d!\frac{t!}{d!(t-d)!(t+1-d)}\right) \\
&= \log\left((d-1)!\binom{t}{d-1}\right).
\end{aligned}$$

□

5. The union-closed conjecture

Definition 5.1 Let $\mathcal{A}$ be a finite family of sets. $\mathcal{A}$ is **union-closed** if $A \cup B \in \mathcal{A}$ for all $A, B \in \mathcal{A}$.

Conjecture 5.2 (Union-closed Conjecture) If $\mathcal{A}$ is a non-empty union-closed family, then there exists x that belongs to at least $\frac{1}{2}|\mathcal{A}|$ sets in $\mathcal{A}$.

Theorem 5.3 (Gilmer) There exists a constant $c > 0$ such that if $\mathcal{A}$ is any union-closed family, then there exists x that belongs to at least $c|\mathcal{A}|$ of the sets in $\mathcal{A}$.

Example 5.4 Let $\mathcal{A} = [n]^{(pn)} \cup [n]^{((\geq(2p-p^2-o(1))n)}$. Then with high probability, if A, B are random elements of $[n]^{(pn)}$, then $|A \cup B| \geq (2p - p^2 - o(1))n$ (since the intersect is likely of size at most $p^2 n$). If $1 - (2p - p^2 - o(1)) = p$, then almost all of $\mathcal{A}$ is contained in $[n]^{(pn)}$. The solutions of p occur roughly when $1 - 3p + p^2 = 0$, which has solutions $p = \frac{1}{2}\left(3 \pm \sqrt{5}\right)$.

If we want to prove Gilmer, it is natural to let A, B be independent uniformly random elements of $\mathcal{A}$ and to consider $H(A \cup B)$. Since $\mathcal{A}$ is union-closed, $A \cup B \in \mathcal{A}$, so $H(A \cup B) \leq \log|\mathcal{A}|$. Now we would like to get a lower bound for $H(A \cup B)$ assuming that no x belongs to more than $p|\mathcal{A}|$ sets in $\mathcal{A}$.

Lemma 5.5 Suppose $c > 0$ is such that $h(xy) \geq c(xh(y) + yh(x))$ for every $x, y \in [0, 1]$. Let $\mathcal{A}$ be a family of sets such that every element of $\cup\mathcal{A}$ belongs to fewer than $p|\mathcal{A}|$ members of $\mathcal{A}$. Let A, B be independent uniformly members of $\mathcal{A}$. Then

$$H(A \cup B) > c(1 - p)(H(A) + H(B)).$$

Proof (Hints).
- Think of A, B as characteristic functions. Write $A_{<k}$ for $(A_1, ..., A_{k-1})$.
- Explain why it is enough to prove that $H((A \cup B)_k \mid A_{<k}, B_{<k}) > c(1 - p)\left(H(A_k \mid A_{<k}) + H\left(B_k \mid H_{B_{<k}}\right)\right)$ for all k.
- For each $u, v \in \{0, 1\}^{k-1}$, write $p(u) = \mathbb{P}(A_k = 0 \mid A_{<k} = u)$ and $q(v) = \mathbb{P}(B_k = 0 \mid B_{<k} = v)$. Find a (simple) expression for $H((A \cup B)_k \mid A_{<k} = u, B_{<k} = v)$.
- Expand $H((A \cup B)_k \mid A_{<k}, B_{<k})$, give an upper bound, then simplify it (hint: law of total probability).

□

Proof. Think of A, B as characteristic functions. Write $A_{<k}$ for $(A_1, ..., A_{k-1})$. By the Chain Rule, it is enough to prove for every k that

$$H((A \cup B)_k \mid (A \cup B)_{<k}) > c(1 - p)\left(H(A_k \mid A_{<k}) + H\left(B_k \mid H_{B_{<k}}\right)\right).$$

By Lemma 1.25,

$$H((A \cup B)_k \mid (A \cup B)_{<k}) \geq H((A \cup B)_k \mid A_{<k}, B_{<k})$$

For each $u, v \in \{0, 1\}^{k-1}$, write $p(u) = \mathbb{P}(A_k = 0 \mid A_{<k} = u)$ and $q(v) = \mathbb{P}(B_k = 0 \mid B_{<k} = v)$. Then, since A and B are independent,

$$H((A \cup B)_k \mid A_{<k} = u, B_{<k} = v)$$

$$= -\sum_{i=0}^{1} \mathbb{P}((A \cup B)_k = i \mid A_{<k} = u, B_{<k} = v) \log \mathbb{P}((A \cup B)_k = i \mid A_{<k} = u, B_{<k} = v)$$
$$= h(p(u)q(v)).$$

which by hypothesis is at least $c(p(u)h(q(v)) + q(v)h(p(u)))$. So

$$\begin{aligned} H((A \cup B)_k \mid (A \cup B)_{<k}) &\geq c \sum_{u,v} \mathbb{P}(A_{<k} = u)\mathbb{P}(B_{<k} = v)(p(u)h(q(v)) + q(v)h(p(u))) \\ &= c \cdot \sum_u \mathbb{P}(A_{<k} = u)p(u) \cdot \sum_v \mathbb{P}(B_{<k} = v)h(q(v)) \\ &\quad + c \cdot \sum_u \mathbb{P}_{A_{<k}=u} h(p(u)) \cdot \sum_v \mathbb{P}(B_{<k} = v)q(v) \end{aligned}$$

But by law of total probability,

$$\sum_u \mathbb{P}(A_{<k} = u)\mathbb{P}(A_k = 0 \mid A_{<k} = u) = \mathbb{P}(A_k = 0),$$

and

$$\sum_v \mathbb{P}(B_{<k} = v)h(q(v)) = \sum_v \mathbb{P}(B_{<k} = v)H(B_k \mid B_{<k} = v) = H(B_k \mid B_{<k})$$

Similarly for the other term, so the RHS of the inequality equals

$$c(\mathbb{P}(A_k = 0)H(B_k \mid B_{<k}) + \mathbb{P}(B_k = 0)H(A_k \mid A_{<k})),$$

which by hypothesis (since $\mathbb{P}(A_k = 0) = \mathbb{P}(B_k = 0) > 1 - p$) is greater than

$$c(1 - p)(H(A_k \mid A_{<k}) + H(B_k \mid B_{<k}))$$

as required. □

Corollary 5.6 Let $\mathcal{A}$, p and c be as in Lemma 5.5. If $\mathcal{A}$ is union-closed, then we must have $p \geq 1 - 1/2c$.

Proof (Hints). Straightforward. □

Proof. Let A and B be independent uniformly random elements of $\mathcal{A}$. Since $\mathcal{A}$ is union-closed, $A \cup B \in \mathcal{A}$, so $H(A \cup B) \leq \log|\mathcal{A}|$. Also, $H(A) = H(B) = \log|\mathcal{A}|$. Hence, by Lemma 5.5, $2c(1 - p) \leq 1$. □

Corollary 5.6 gives a non-trivial bound as long as $c > 1/2$. We shall obtain $1/\left(\sqrt{5} - 1\right)$.

We start by proving the diagonal case, i.e. where $x = y$.

Lemma 5.7 (Boppana) For every $x \in [0, 1]$,

$$h(x^2) \geq \varphi \cdot x \cdot h(x),$$

where $\varphi = \frac{1}{2}\left(\sqrt{5} + 1\right)$.

Proof (Hints).

- Let $\psi = 1/\varphi$. Show that equality holds when $x = \psi, 0, 1$.

- Let $f(x) = h(x^2) - \varphi \cdot x \cdot h(x)$. Show that $f'''(x) = 0$ iff $-\varphi x^3 - 4x^2 + 3\varphi x - 4 + 2\varphi = 0$. (Advice: use natural logs and find expressions for $h'(x)$, $h''(x)$ and $h'''(x)$ first).
- Explain why f''' has at most two roots in $(0,1)$ and so f has at most five roots in $[0,1]$.
- Show that f has a double root at 0 and at ψ.
- Explain why f must have constant sign on $[0,1]$, and by considering small x, show that there is x with $f(x) > 0$.

$\square$

Proof. Write $\psi = 1/\varphi = \frac{1}{2}\left(\sqrt{5}-1\right)$. Then $\psi^2 = 1 - \psi$. So $h(\psi^2) = h(1-\psi) = h(\psi)$ and $\varphi\psi = 1$, so $h(\psi^2) = \varphi \cdot \psi \cdot h(\psi)$. So equality holds when $x = \psi$, and also when $x = 0, 1$.

Toolkit: $\ln(2) \cdot h(x) = -x \ln x - (1-x)\ln(1-x)$. Then

$$\ln(2) \cdot h'(x) = -\ln x - 1 + \ln(1-x) + 1 = \ln(1-x) - \ln(x)$$

and

$$\ln(2) \cdot h''(x) = -\frac{1}{x} - \frac{1}{1-x} = -\frac{1}{x(1-x)}$$

and

$$\ln(2) \cdot h'''(x) = \frac{1}{x^2} - \frac{1}{(1-x)^2} = \frac{1-2x}{x^2(1-x)^2}.$$

Let $f(x) = h(x^2) - \varphi \cdot x \cdot h(x)$. Then

$$\begin{aligned}
f'(x) &= 2xh'(x^2) - \varphi h(x) - \varphi x h'(x) \\
f''(x) &= 2h'(x^2) + 4x^2 h''(x^2) - 2\varphi h'(x) - \varphi x h''(x) \\
f'''(x) &= 4xh''(x^2) + 8xh''(x^2) + 8x^3 h'''(x^2) - 3\varphi h''(x) - \varphi x h'''(x) \\
&= 12xh''(x^2) + 8x^3 h'''(x^2) - 3\varphi h''(x) - \varphi x h'''(x)
\end{aligned}$$

So

$$\begin{aligned}
\ln(2) f'''(x) &= \frac{-12x}{x^2(1-x^2)} + \frac{8x^3(1-2x^2)}{x^4(1-x^2)^2} + \frac{3\varphi}{x(1-x)} - \frac{\varphi x(1-2x)}{x^2(1-x)^2} \\
&= \frac{-12}{x(1-x^2)} + \frac{8(1-2x^2)}{x(1-x^2)^2} + \frac{3\varphi}{x(1-x)} - \frac{\varphi(1-2x)}{x(1-x)^2} \\
&= \frac{-12(1-x^2) + 8(1-2x^2) + 3\varphi(1-x)(1+x)^2 - \varphi(1-2x)(1+x)^2}{x(1-x)^2(1+x)^2}
\end{aligned}$$

which is zero iff

$$-12 + 12x + 8 - 16x^2 + 3\varphi(1 + x - x^2 - x^3) - \varphi(1 - 3x^2 - 2x^3)$$

$$= -\varphi x^3 - 4x^2 + 3\varphi x - 4 + 2\varphi = 0.$$

So the numerator of $f'''(x)$ is a cubic with negative leading coefficient and constant term, so it has a negative root, so it has at most two roots in $(0, 1)$. It follows (by Rolle's theorem) that f has at most five roots in $[0, 1]$, up to multiplicity. But

$$f'(x) = 2x(\log(1-x^2) - \log(x^2)) + \varphi(x\log x + (1-x)\log(1-x)) - \varphi x(\log(1-x) - \log x)$$

So $f'(0) = 0$, so f has a double root at 0. Now

$$\begin{aligned} f'(\psi) &= 2\psi(\log\psi - 2\log\psi) + \varphi(\psi\log\psi + 2(1-\psi)\log\psi) - (2\log\psi - \log\psi) \\ &= -2\psi\log\psi + \log\psi + 2\varphi\log\psi - 2\log\psi \\ &= 2\log\psi(-\psi + \varphi - 1) \\ &= 2\varphi\log\psi(-\psi^2 - 1 - \psi) = 0 \end{aligned}$$

So there is a double root at ψ. Also, $f(1) = 0$. So f is either non-negative on all of $[0, 1]$ or non-positive on all of $[0, 1]$. If x is small,

$$\begin{aligned} f(x) &= x^2\log\frac{1}{x^2} + (1-x^2)\log\frac{1}{1-x^2} - \varphi x\left(x\log\frac{1}{x} + (1-x)\log\frac{1}{1-x}\right) \\ &= 2x^2\log\frac{1}{x} - \varphi x^2\log\frac{1}{x} + O(x^2). \end{aligned}$$

So, because $2 > \varphi$, there exists x such that $f(x) > 0$. $\square$

Lemma 5.8 The function $f(x, y) = \frac{h(xy)}{xh(y)+yh(x)}$ is minimised on $(0, 1)^2$ at a point where $x = y$.

Proof (Hints).

- Show that we can extend f continuously to the boundary by setting $f(x, y) = 1$ whenever x or y is 0 or 1 (for the case when x or y tend to 0 separately, consider an expansion for xy small, and for the case when x and y tend to 1, consider when one of x or y is 1).
- Pick any point in $(0, 1)^2$ to show that f is minimised somewhere in that region.
- Let (x^*, y^*) be a minimum with $f(x^*, y^*) = \alpha$. Let $g(x) = h(x)/x$.
- By considering the expression $g(xy) - \alpha(g(x) + g(y))$ and partial derivatives, show that $x^*g'(x^*) = y^*g'(y^*)$.
- Show that $xg'(x)$ is an injection by considering its derivative.

$\square$

Proof. We can extend f continuously to the boundary by setting $f(x, y) = 1$ whenever x or y is 0 or 1. To see this, note first that it is easy if neither x nor y is 0. If either x or y is small then $h(xy) = -xy(\log x + \log y) + O(xy)$, and

$$\begin{aligned} xh(y) + yh(x) &= -x(y\log y + O(y)) - y(x\log x + O(x)) \\ &= h(xy) \quad \text{up to } O(xy) \end{aligned}$$

So it tends to 1 again.

We can check that $f(1/2,1/2) < 1$, so f is minimised somewhere in $(0,1)^2$. Let (x^*,y^*) be a minimum with $f(x^*,y^*) = \alpha$. For convenience, let $g(x) = h(x)/x$ and note that $f(x,y) = \frac{g(xy)}{g(x)+g(y)}$. Also, $g(xy) - \alpha(g(x)+g(y)) \geq 0$ with equality at (x^*,y^*). So the partial derivatives of the LHS are both 0 at (x^*,y^*):

$$\begin{aligned}y^*g'(x^*y^*) - \alpha g'(x^*) &= 0\\ x^*g'(x^*y^*) - \alpha g'(y^*) &= 0.\end{aligned}$$

So $x^*g'(x^*) = y^*g'(y^*)$. So it is enough to prove that $xg'(x)$ is an injection. We have

$$g'(x) = \frac{h'(x)}{x} - \frac{h(x)}{x^2}$$

so

$$\begin{aligned}xg'(x) &= h'(x) - \frac{h(x)}{x}\\ &= \log(1-x) - \log x + \frac{x\log x + (1-x)\log(1-x)}{x}\\ &= \frac{\log(1-x)}{x}.\end{aligned}$$

Differentiating gives

$$-\frac{1}{x(1-x)} - \frac{\log(1-x)}{x^2} = \frac{-x-(1-x)\log(1-x)}{x^2(1-x)}$$

The numerator differentiates to $-1+1+\log(1-x)$ which is negative. Also, it equals 0 at 0, so it has a constant sign. Thus, $xg'(x)$ is indeed an injection. □

Combining this with Boppana we get that

$$h(xy) \geq \frac{\varphi}{2}(xh(y) + yh(x))$$

This allows us to take $p = 1 - \frac{1}{\varphi} = \frac{3-\sqrt{5}}{2}$.

6. Entropy in additive combinatorics

We shall need two "simple" results from additive combinatorics due to Imre Ruzsa.

Definition 6.1 Let G be an abelian group and let $A, B \subseteq G$. The **sumset** $A+B$ of A and B is the set

$$\{x+y : x \in A, y \in B\}$$

and the **difference set** $A-B$ is the set

$$\{x-y : x \in A, y \in B\}.$$

Write $2A$ for $A+A$, $3A$ for $A+A+A$, etc.

Definition 6.2 The **Ruzsa distance** $d(A,B)$ is

$$\frac{|A-B|}{|A|^{1/2}\cdot|B|^{1/2}}.$$

Lemma 6.3 (Ruzsa Triangle Inequality) $d(A,C) \le d(A,B)\cdot d(B,C)$.

Proof (Hints). Expand the stated inequality and consider an appropriate injection. □

Proof. This is equivalent to the statement

$$|A-C|\cdot|B| \le |A-B|\cdot|B-C|.$$

For each $x \in A-C$, pick $a(x) \in A$, $c(x) \in C$ such that $x = a(x) - c(x)$. Define the map

$$\begin{aligned}\varphi : (A-C)\times B &\to (A-B)\times(B-C),\\ (x,b) &\mapsto (a(x)-b, b-c(x)).\end{aligned}$$

Adding the coordinates of $\varphi(x,b)$ gives x, so we can calculate $a(x)$ and $c(x)$ from $\varphi(x,b)$, and hence b. So φ is an injection. □

Lemma 6.4 (Ruzsa Covering Lemma) Let G be an abelian group and let $A, B \subseteq G$ be finite. Then A can be covered by at most $|A+B|/|B|$ translates of $B-B$.

Proof (Hints). Consider a maximal subset $\{x_1, ..., x_k\} \subseteq A$ such that the $x_i + B$ are disjoint. □

Proof. Let $\{x_1, ..., x_k\}$ be a maximal subset of A such that the sets $x_i + B$ are disjoint. Then for all, $a \in A$, there exists i such that $(a+B)\cap(x_i+B) \ne \emptyset$, i.e. $a \in (x_i + (B-B))$. So A can be covered by k translates of $B-B$. But since the x_i+B are disjoint,

$$|B|k = |\{x_1, ..., x_k\} + B| \le |A+B|.$$

□

Let X, Y be discrete random variables taking values in an abelian group. What is $X+Y$ when X and Y are independent? For each z, $\mathbb{P}(X+Y=z) = \sum_{x+y=z}\mathbb{P}(X=x)\mathbb{P}(Y=y)$. Writing p_x and q_y for $\mathbb{P}(X=x)$ and $\mathbb{P}(Y=y)$, this gives

$$\sum_{x+y=z} p_x p_y = (p*q)(z)$$

where $p(x) = p_x$, $q(y) = q_y$. So sums of independent random variables correspond to convolutions.

Definition 6.5 Let G be an abelian group and let X, Y be G-valued random variables. The **(entropic) Ruzsa distance** between X and Y is

$$\begin{aligned}d(X;Y) &= H(X'-Y') - \frac{1}{2}H(X) - \frac{1}{2}H(Y)\\ &= H(X'-Y') - \frac{1}{2}H(X') - \frac{1}{2}H(Y').\end{aligned}$$

where X', Y' are independent copies of X, Y.

Lemma 6.6 If A, B are finite subsets of G and X, Y are uniform on A, B respectively, then

$$d(X;Y) \leq \log d(A, B).$$

Proof (Hints). Straightforward. □

Proof. WLOG X, Y are independent. Then

$$\begin{aligned} d(X,Y) &= H(X-Y) - \frac{1}{2}H(X) - \frac{1}{2}H(Y) \\ &\leq \log|A-B| - \frac{1}{2}\log|A| - \frac{1}{2}\log|B| = \log d(A, B). \end{aligned}$$

□

Lemma 6.7 Let X, Y be G-valued random variables. Then

$$H(X-Y) \geq \max\{H(X), H(Y)\} - I(X:Y).$$

Proof (Hints). Use that $H(X-Y) \geq H(X-Y \mid Y)$ and $H(X-Y) \geq H(X-Y \mid X)$. □

Proof. We have

$$\begin{aligned} H(X-Y) &\geq H(X-Y \mid Y) \text{ by Subadditivity} \\ &= H(X-Y, Y) - H(Y) \\ &= H(X, Y) - H(Y) \text{ by Invariance} \\ &= H(X) + H(Y) - H(Y) - I(X:Y) \\ &= H(X) - I(X:Y). \end{aligned}$$

We use Invariance with the bijection $(x, y) \mapsto (x-y, y)$. By symmetry, we also have $H(X-Y) \geq H(Y) - I(X:Y)$. □

Corollary 6.8 If X, Y are G-valued RVs, then $d(X;Y) \geq 0$.

Proof (Hints). Straightforward. □

Proof. WLOG X and Y are independent. Then $I(X:Y) = 0$, so $H(X-Y) \geq \max\{H(X), H(Y)\} \geq \frac{1}{2}(H(X) + H(Y))$. □

Lemma 6.9 If X, Y are G-valued RVs, then $d(X;Y) = 0$ iff there is some (finite) subgroup H of G such that X and Y are uniform on cosets of H.

Proof (Hints).

- $\Longleftarrow$: straightforward.
- $\Longrightarrow$: assume WLOG that X and Y are independent. By considering entropy, explain why $X-Y$ and Y are independent.
- Deduce that for X supported on A and Y supported on B, for all $z \in A-B$ and $y_1, y_2 \in B$, $\mathbb{P}(X = y_1 + z) = \mathbb{P}(X = y_2 + z)$, and show that this implies that $z + B \subseteq A$.

- Deduce that $A = B + z$ for all $z \in A - B$, and so that $A - x$ is constant over $x \in A$.
- Deduce that $A - A$ is a subgroup.

$\square$

Proof. $\Longleftarrow$: If X, Y are uniform on $x + H, y + H$ then $X' - Y'$ is uniform on $(x - y) + H$, so $H(X' - Y') = H(X) = H(Y)$.

$\Longrightarrow$: WLOG X and Y are independent. We have $H(X - Y) = \frac{1}{2}(H(X) + H(Y))$. So equality must hold throughout the proof of Lemma 6.7 and Corollary 6.8, thus $H(X - Y \mid Y) = H(X - Y)$. Therefore, $X - Y$ and Y are independent. So for every $z \in A - B$ and $y_1, y_2 \in B$,

$$\mathbb{P}(X - Y = z \mid Y = y_1) = \mathbb{P}(X - Y = z \mid Y = y_2),$$

where $A = \{x : \mathbb{P}(X = x) \neq 0\}$ and $B = \{y : \mathbb{P}(Y = y) \neq 0\}$. We can write this as

$$\mathbb{P}(X = y_1 + z) = \mathbb{P}(X = y_2 + z)$$

So $\mathbb{P}(X = x)$ is constant on $z + B$. In particular, $z + B \subseteq A$ ($\mathbb{P}(X = x)$ must be non-zero on $z + B$, as otherwise $(z + B) \cap A = \emptyset$, i.e. $z \notin A - B$). By the same argument, $A - z \subseteq B$. So $A = B + z$ for all $z \in A - B$. So for every $x \in A$ and $y \in B$, $A = B + x - y$, so $A - x = B - y$. Hence, $A - x$ is the same for every $x \in A$. Therefore, $A - x = \cup_{x \in A} (A - x) = A - A$ for all $x \in A$. It follows that

$$A - A + A - A = (A - A) - (A - A) = A - x - (A - x) = A - A.$$

So $A - x = A - A$ is a subgroup, and so A is a coset of $A - A$. $B = A + x$, so B is also a coset of $A - A$. Also, as stated above, X is uniform on $z + B = A$ and Y is uniform on $A - z = B$. $\square$

Lemma 6.10 (Entropic Ruzsa Triangle Inequality) Let X, Y, Z be G-valued random variables. Then $d(X; Z) \leq d(X; Y) + d(Y; Z)$.

Proof (Hints). Simplify the desired inequality and use Lemma 1.26 (where $X - Z$ depends on two different (pairs of) random variables). $\square$

Proof. We must show (assuming WLOG that X, Y, Z are independent) that

$$\begin{aligned} &H(X - Z) - \frac{1}{2}H(X) - \frac{1}{2}H(Z) \\ &\leq H(X - Y) - \frac{1}{2}H(X) - \frac{1}{2}H(Y) + H(Y - Z) - \frac{1}{2}H(Y) - \frac{1}{2}H(Z), \end{aligned}$$

i.e. that $H(X - Z) + H(Y) \leq H(X - Y) + H(Y - Z)$. Since $X - Z$ depends on $(X - Y, Y - Z)$ and on (X, Z), by Lemma 1.26,

$$H(X - Y, Y - Z, X, Z) + H(X - Z) \leq H(X - Y, Y - Z) + H(X, Z)$$

i.e. $H(X, Y, Z) + H(X - Z) \leq H(X, Z) + H(X - Y, Y - Z)$. By independence and Subadditivity, we get $H(X - Z) + H(Y) \leq H(X - Y) + H(Y - Z)$. $\square$

Lemma 6.11 (Submodularity for Sums) If X, Y, Z are independent G-valued RVs, then

$$H(X+Y+Z)+H(Z) \leq H(X+Z)+H(Y+Z).$$

Proof (Hints). Use Lemma 1.26. ☐

Proof. $X+Y+Z$ is a function of $(X+Z, Y)$ and of $(X, Y+Z)$. Therefore, by Lemma 1.26,

$$H(X+Z, Y, X, Y+Z)+H(X+Y+Z) \leq H(X+Z, Y)+H(X, Y+Z),$$

thus $H(X, Y, Z)+H(X+Y+Z) \leq H(X+Z)+H(Y)+H(X)+H(Y+Z)$. By independence and cancelling equal terms, we get the desired inequality. ☐

Lemma 6.12 Let G be an abelian group and let X be a G-valued random variable. Then $d(X;-X) \leq 2d(X;X)$.

Proof (Hints). Consider independent copies X_1, X_2, X_3 of X, use Lemma 6.7. ☐

Proof. Let X_1, X_2, X_3 be independent copies of X. Then by Lemma 6.7,

$$\begin{aligned}
d(X;-X) &= H(X_1+X_2) - \frac{1}{2}H(X_1) - \frac{1}{2}H(X_2) \\
&\leq H(X_1+X_2-X_3) - H(X) \\
&\leq H(X_1-X_3) + H(X_2-X_3) - H(X_3) - H(X) \\
&= 2d(X;X)
\end{aligned}$$

by Submodularity for Sums and since X_1, X_2, X_3 are all copies of X. ☐

Corollary 6.13 Let X and Y be G-valued random variables. Then $d(X;-Y) \leq 5d(X;Y)$.

Proof (Hints). Straightforward. ☐

Proof. By the Entropic Ruzsa Triangle Inequality,

$$\begin{aligned}
d(X;-Y) &\leq d(X;Y) + d(Y;-Y) \\
&\leq d(X;Y) + 2d(Y;Y) \\
&\leq d(X;Y) + 2(d(Y;X)+d(X;Y)) = 5d(X;Y).
\end{aligned}$$

☐

Definition 6.14 Let X, Y, U, V be G-valued random variables. The **conditional distance** is

$$d(X \mid U; Y \mid V) = \sum_{u,v} \mathbb{P}(U=u)\mathbb{P}(V=v) d(X \mid U=u; Y \mid V=v).$$

Definition 6.15 Let X, Y, U be G-valued random variables. The **simultaneous conditional distance** of X to Y given U is

$$d(X;Y \parallel U) := \sum_u \mathbb{P}(U=u) d(X \mid U=u; Y \mid U=u).$$

Definition 6.16 We say that X', Y' are **conditionally independent trials** of X, Y given U if X' is distributed like X, Y' like Y, and for each u, $X' \mid U = u$ is distributed like $X \mid U = u$, $Y' \mid U = u$ is distributed like $Y \mid U = u$, and $X' \mid U = u$ and $Y' \mid U = u$ are independent.

In that case, $d(X; Y \parallel U) = H(X' - Y' \mid U) - \frac{1}{2}H(X' \mid U) - \frac{1}{2}H(Y' \mid U)$.

Lemma 6.17 (Entropic BSG Theorem) Let A, B be G-valued RVs. Then

$$d(A; B \parallel A + B) \leq 3I(A : B) + 2H(A + B) - H(A) - H(B).$$

Proof (Hints).

- Let A', B' be conditionally independent trials of A, B given $A + B$.
- Show that $H(A' \mid A + B) = H(A) + H(B) - I(A : B) - H(A + B)$.
- Let (A_1, B_1) and (A_2, B_2) be conditionally independent trials of (A, B) given $A + B$.
- Explain why $H(A_1 - B_2) \leq H(A_1 - B_2, A_1) + H(A_1 - B_2, B_1) - H(A_1 - B_2, A_1, B_1)$.
- Use that $A_1 + B_1 = A_2 + B_2$ to bound each of the first two terms on the RHS of the above, and rewrite the $H(A_1 - B_2, A_1, B_1)$ term, using the conditional independence of (A_1, B_1) and (A_2, B_2), to conclude the result.

□

Proof. We have

$$d(A, B \parallel A + B) = H(A' - B' \mid A + B) - \frac{1}{2}H(A' \mid A + B) - \frac{1}{2}H(B' \mid A + B),$$

where A', B' are conditionally independent trials of A, B given $A + B$. Now

$$\begin{aligned} H(A' \mid A + B) &= H(A \mid A + B) = H(A, A + B) - H(A + B) \\ &= H(A, B) - H(A + B) \\ &= H(A) + H(B) - I(A : B) - H(A + B). \end{aligned}$$

Similarly, $H(B' \mid A + B) = H(A) + H(B) - I(A : B) - H(A + B)$, so

$$\frac{1}{2}H(A' \mid A + B) + \frac{1}{2}H(B' \mid A + B)$$

is also the same. By Subadditivity, $H(A' - B' \mid A + B) \leq H(A' - B')$. Let (A_1, B_1) and (A_2, B_2) be conditionally independent trials of (A, B) given $A + B$ (here, A_1 plays the role of A', B_2 plays the role of B', and each comes with another RV since we know the value of $A + B$). Then $H(A' - B') = H(A_1 - B_2)$. By Submodularity,

$$H(A_1 - B_2) \leq H(A_1 - B_2, A_1) + H(A_1 - B_2, B_1) - H(A_1 - B_2, A_1, B_1)$$

Also,

$$H(A_1 - B_2, A_1) = H(A_1, B_2) \leq H(A_1) + H(B_2) = H(A) + H(B)$$

and since $A_1 + B_1 = A_2 + B_2$,

$$H(A_1 - B_2, B_1) = H(A_2 - B_1, B_1) = H(A_2, B_1) \leq H(A) + H(B).$$

Finally, since $A_1 + B_1 = A_2 + B_2$,

$$\begin{aligned} H(A_1 - B_2, A_1, B_1) &= H(A_1, B_1, A_2, B_2) \\ &= H(A_1, B_1, A_2, B_2 \mid A + B) + H(A + B) \\ &= 2H(A, B \mid A + B) + H(A + B) \\ &= 2H(A, B) - H(A + B) \\ &= 2H(A) + 2H(B) - 2I(A : B) - H(A + B). \end{aligned}$$

where the third line is by conditional independence of (A_1, B_1) and (A_2, B_2). Adding or subtracting as appropriate all these terms gives the required inequality. □

7. A proof of Marton's conjecture in $\mathbb{F}_2^n$

We shall prove the following theorem.

Theorem 7.1 (Green, Manners, Tao, Gowers) There is a polynomial p with the following property: if $n \in \mathbb{N}$ and $A \subseteq \mathbb{F}_2^n$ is such that $|A + A| \leq C|A|$, then there is a subspace $H \subseteq \mathbb{F}_2^n$ of size at most $|A|$ such that A is contained in the union of at most $p(C)$ translates of H. Equivalently, there exists $K \subseteq \mathbb{F}_2$, $|K| \leq p(C)$, such that $A \subseteq K + H$.

In fact, we shall prove the following statement:

Theorem 7.2 (EPFR) Let $G = \mathbb{F}_2^n$. There is an absolute constant α with the following property:

Let X, Y be G-valued random variables. Then there exists a subgroup H of G such that

$$d(X; U_H) + d(U_H; Y) \leq \alpha d(X; Y),$$

where U_H is a random variable distributed uniformly on H.

Lemma 7.3 Let X be a discrete random variable and write $p_x = \mathbb{P}(X = x)$. Then there exists x such that $p_x \geq 2^{-H(X)}$.

Proof (Hints). By contradiction. □

Proof. If not, then $H(X) = \sum_x p_x \log(1/p_x) > H(X) \sum_x p_x = H(X)$: contradiction. □

Proposition 7.4 EPFR implies Green, Manners, Tao, Gowers.

Proof (Hints).

- Let $A \subseteq \mathbb{F}_2^n$ and $|A + A| \leq C|A|$. Let U_H be uniformly distributed on H, let X and Y be independent copies of U_A. Show that $d(X; U_H) \leq \frac{1}{2}\alpha \log C$.
- Deduce that there exists z such that

  $$\mathbb{P}(X + U_H = z) \geq |A|^{-1/2}|H|^{-1/2}C^{-\alpha/2}$$

  and find an expression for the LHS.
- Let $B = A \cap (z + H)$. Show that A can be covered by at most $\frac{|A+B|}{|B|}$ translates of H.

- Use that $B \subseteq A, z+H$ to show that
$$\frac{|A+B|}{|B|} \le C^{\alpha/2+1}\frac{|A|^{1/2}}{|H|^{1/2}} \le C^{\alpha+1}.$$
- Consider the cases $|H| \le |A|$ and $|H| > |A|$: if the latter, then consider a subgroup H' of H of size between $|A|/2$ and $|A|$ (why does this exist?).

$\square$

Proof. Let $A \subseteq \mathbb{F}_2^n$ and $|A+A| \le C|A|$. Let X and Y be independent copies of U_A. Then by EPFR, there exists a subgroup H such that $d(X;U_H) + d(U_H;X) \le \alpha d(X;Y)$, so $d(X;U_H) \le \frac{\alpha}{2} d(X;Y)$. But since we are in $\mathbb{F}_2^n$,
$$\begin{aligned} d(X;Y) &= H(U_A - U_A') - \frac{1}{2}H(U_A) - \frac{1}{2}H(U_A') = H(U_A + U_A') - H(U_A) \\ &\le \log C|A| - \log|A| = \log C, \end{aligned}$$
by Maximality. So $d(X;U_H) \le \frac{1}{2}\alpha \log C$, i.e.
$$\begin{aligned} H(X+U_H) &\le \frac{1}{2}H(X) + \frac{1}{2}H(U_H) + \frac{1}{2}\alpha\log C \\ &= \frac{1}{2}\log|A| + \frac{1}{2}\log|H| + \frac{1}{2}\alpha\log C. \end{aligned}$$
Therefore by Lemma 7.3, there exists z such that
$$\mathbb{P}(X+U_H = z) \ge |A|^{-1/2}|H|^{-1/2}C^{-\alpha/2}.$$
But $\mathbb{P}(X+U_H=z) = \frac{A\cap(z-H)}{|A||H|} = \frac{A\cap(z+H)}{|A||H|}$. So there exists $z \in G$ such that
$$|A\cap(z+H)| \ge C^{-\alpha/2}|A|^{-1/2}|H|^{-1/2}.$$
Let $B = A\cap(z+H)$. Let $B = A\cap(z+H)$. By Ruzsa Covering Lemma, we can cover A by at most $\frac{|A+B|}{|B|}$ translates of $B-B = B+B$. But $B \subseteq z+H$ so $B+B \subseteq 2z+H+H = H$. So A can be covered by at most $\frac{|A+B|}{|B|}$ translates of H. But since $B \subseteq A$, $|A+B| \le |A+A| \le C|A|$. So
$$\frac{|A+B|}{|B|} \le \frac{C|A|}{C^{-\alpha/2}|A|^{1/2}|H|^{1/2}} = C^{\alpha/2+1}\frac{|A|^{1/2}}{|H|^{1/2}}.$$
Since B is contained in $z+H$, $|H| \ge C^{-\alpha/2}|A|^{1/2}|H|^{1/2}$, which implies $|H| \ge C^{-\alpha}|A|$. So
$$C^{\alpha/2+1}\frac{|A|^{1/2}}{|H|^{1/2}} \le C^{\alpha+1}.$$
If $|H| \le |A|$, then we are done (with polynomial $p(x) = x^{\alpha+1}$). Otherwise, since $B \subseteq A$, $|A| \ge C^{-\alpha/2}|A|^{1/2}|H|^{1/2}$, which implies $|H| \le C^{\alpha}|A|$. Pick a subgroup H' of H of size between $|A|/2$ and $|A|$. Then H is a union of $|H|/|H'| \le 2C^{\alpha}$ translates of H', so A is a union of at most $2C^{2\alpha+1}$ translates of H'. $\square$

Now we reduce further. We shall prove the following statement.

Theorem 7.5 (EPFR') There is an absolute constant $\eta > 0$ such that if X and Y are any two $\mathbb{F}_2^n$-valued RVs, with $d(X;Y) > 0$, then there exist $\mathbb{F}_2^n$-valued RVs U and V such that

$$\tau_{X,Y}(U;V) := d(U;V) + \eta(d(U;X) + d(V;Y)) < d(X;Y).$$

Proposition 7.6 EPFR' with constant η implies EPFR with constant $1/\eta$.

Proof (Hints).

- By compactness, we can find $\mathbb{F}_2^n$-valued RVs U, V such that $\tau_{X,Y}(U;V)$ is minimised.
- Assuming that $d(U;V) \neq 0$, use the Ruzsa Triangle Inequality to derive a contradiction.
- Conclude using Lemma 6.9.

$\square$

Proof. By compactness, we can find $\mathbb{F}_2^n$-valued RVs U, V such that $\tau_{X,Y}(U;V)$ is minimised. If $d(U;V) \neq 0$, then by EPFR', there exist $\mathbb{F}_2^n$-valued RVs Z, W such that $\tau_{UV}(Z;W) < d(U;V)$. But then by the Ruzsa Triangle Inequality,

$$\begin{aligned}
\tau_{X,Y}(Z;W) &= d(Z;W) + \eta(d(Z;X) + d(W;Y)) \\
&\le d(Z;W) + \eta(d(Z;U) + d(W;V)) + \eta(d(U;X) + d(V;Y)) \\
&< d(U;V) + \eta(d(U;X) + d(V;Y)) \\
&= \tau_{X,Y}(U;V),
\end{aligned}$$

which is a contradiction. It follows that $d(U;V) = 0$. So by Lemma 6.9, there exists H such that U and V are uniform on cosets of H, so

$$\eta(d(U;X) + d(V;Y)) = \eta(d(U_H;X) + d(U_H;Y)) < d(X;Y),$$

since $d(\cdot;\cdot)$ is invariant under constant shifts of either of its arguments. This gives EPFR with constant $1/\eta$. $\square$

Notation 7.7 Write $\tau_{X,Y}(U \mid Z; V \mid W)$ for $\sum_{z,w} \mathbb{P}(Z = z)\mathbb{P}(W = w)\tau_{X,Y}(U \mid Z = z; V \mid W = w)$ and $\tau_{X,Y}(U;V \parallel Z)$ for $\sum_z \mathbb{P}(Z = z)\tau_{X,Y}(U \mid Z = z; V = Z = z)$.

Remark 7.8 If we can prove EPFR' for conditioned random variables, then by averaging, we get it for some pair of random variables (e.g. of the form $U \mid Z = z$ and $V \mid W = w$).

Lemma 7.9 (Fibring) Let G and H be abelian groups and let $\varphi : G \to H$ be a homomorphism. Let X, Y be G-valued random variables. Then

$$d(X;Y) = d(\varphi(X);\varphi(Y)) + d(X \mid \varphi(X); Y \mid \varphi(Y)) + I(X - Y : (\varphi(X), \varphi(Y)) \mid \varphi(X) - \varphi(Y)).$$

Proof (Hints).

- May assume WLOG that X and Y are independent.
- Use Lemma 1.13 and Additivity.

$\square$

Proof. We may assume WLOG that X and Y are independent. We have

$$\begin{aligned} d(X;Y) &= H(X-Y) - \frac{1}{2}H(X) - \frac{1}{2}H(Y) \\ &= H(\varphi(X)-\varphi(Y)) + H(X-Y \mid \varphi(X)-\varphi(Y)) \\ &-\frac{1}{2}H(\varphi(X)) - \frac{1}{2}H(X \mid \varphi(X)) - \frac{1}{2}H(\varphi(Y)) - \frac{1}{2}H(Y \mid \varphi(Y)) \\ &= d(\varphi(X);\varphi(Y)) + d(X \mid \varphi(X); Y \mid \varphi(Y)) \\ &+H(X-Y \mid \varphi(X)-\varphi(Y)) - H(X-Y \mid \varphi(X),\varphi(Y)) \end{aligned}$$

But the last line equals

$$\begin{aligned} &H(X-Y \mid \varphi(X)-\varphi(Y)) - H(X-Y \mid \varphi(X),\varphi(Y),\varphi(X)-\varphi(Y)) \\ =\ &I(X-Y : (\varphi(X),\varphi(Y)) \mid \varphi(X)-\varphi(Y)). \end{aligned}$$

□

We shall be interested in the following special case.

Corollary 7.10 Let $G = \mathbb{F}_2^n$ and let X_1, X_2, X_3, X_4 be independent G-valued RVs. Then

$$\begin{aligned} d(X_1;X_3) + d(X_2;X_4) &= d((X_1,X_2);(X_3,X_4)) \\ &= d(X_1+X_2; X_3+X_4) + d(X_1 \mid X_1+X_2; X_3 \mid X_3+X_4) \\ &+I(X_1+X_3, X_2+X_4 : X_1+X_2, X_3+X_4 \mid X_1+X_2+X_3+X_4). \end{aligned}$$

Proof (Hints). Straightforward. □

Proof. The first equality is easy to see. For the second, apply Fibring with $X = (X_1, X_2)$, $Y = (X_3, X_4)$ and $\varphi(x,y) = x+y$. □

We shall now set $W = X_1 + X_2 + X_3 + X_4$.

Recall that $d(X;Y \parallel X+Y) \le 3I(X:Y) + 2H(X+Y) - H(X) - H(Y)$. Equivalently, $I(X:Y) \ge \frac{1}{3}(d(X;Y \parallel X+Y) + H(X) + H(Y) - 2H(X+Y))$. Applying this to the mutual information term in Corollary 7.10, we get that it is at least

$$\begin{aligned} &\frac{1}{3}d(X_1+X_3, X_2+X_4; X_1+X_2, X_3+X_4 \parallel X_2+X_3, W) + \frac{1}{3}H(X_1+X_3, X_2+X_4 \mid W) \\ &+\frac{1}{3}H(X_1+X_2, X_3+X_4 \mid W) - \frac{2}{3}H(X_2+X_3, X_2+X_3 \mid W). \end{aligned}$$

which simplifies to

$$\begin{aligned} &\frac{1}{3}d(X_1+X_3, X_2+X_4; X_1+X_2, X_3+X_4 \parallel X_2+X_3, W) \\ &+\frac{1}{3}H(X_1+X_3 \mid W) + \frac{1}{3}H(X_1+X_2 \mid W) - \frac{2}{3}H(X_2+X_3 \mid W) \end{aligned}$$